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Analytical and numerical solutions of the Local Inertial Equations

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ABSTRACT

Neglecting the convective terms in the Saint-Venant Equations (SVE) in flood hydrodynamic modelling can be done without a loss in accuracy of the simulation results. In this case the Local Inertial Equations (LInE) are obtained. Herein we present two analytical solutions for the Local Inertial Equations. The first is the classical instantaneous Dam-Break Problem and the second a steady state solution over a bump. These solutions are compared with two numerical schemes, namely the first order Roe scheme and the second order MacCormack scheme. Comparison between analytical and numerical results shows that the numerical schemes and the analytical solution converge to a unique solution. Furthermore, by neglecting the convective terms the original numerical schemes remain stable without the need for adding entropy correction, artificial viscosity or special initial conditions, as in the case of the full SVE.

1. Introduction

1.1. Governing equations

One-dimensional models are applicable when there is a dominant flow direction or when a more detailed solution is not necessary [1]. The SVE, as presented by Barré de Saint Venant [2], are a well-accepted mathematical description of the physical phenomenon of a 1D free-surface flow [3] based upon the following assumptions:

- The pressure distribution is hydrostatic (the streamlines have a small curvature and vertical acceleration can be neglected).
- The channel bottom slope is small ($\sin(\theta) \approx \theta \wedge \cos(\theta) \approx 1$).
- The flow is one dimensional, assuming uniform velocity across the cross-section.
- Friction and turbulence are introduced by assuming laws applicable to steady state flow.
- Water density is constant.

The equations form a system of coupled non-linear hyperbolic partial differential equations that are described by two dependent variables [3], commonly h (water depth) and v (velocity) but also A (cross section area) or z (water level) and q (unit-discharge), related to two independent variables: x, t (longitudinal direction and time). The system of equations can be further simplified. Assuming a rectangular channel with width 1 m, the equations for the conservative form described by Vázquez-Cendón [4] become

$$\frac{\partial}{\partial t}h+\frac{\partial}{\partial x}q=0 \tag{1}$$

$$\frac{\partial}{\partial t}q+\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right)+\frac{g}{2}\frac{\partial}{\partial x}h^2=gh(S-J) \tag{2}$$

h is the water depth, q the unit-discharge and g the gravitational acceleration, S the bed slope and J the friction slope. Some analytical solutions for simplified cases exist but for practical application numerical methods are preferred [5].

Simplifications of SVE are often sought in order to reduce the computational time or increase the model stability. SVE are frequently simplified into the Kinematic Wave Model, Diffusive wave model and Local Inertial Equations (LInE) Model. Assuming negligible convective terms, the SVE simplify to the LInE. These terms may cause numerical oscillations near discontinuities and wet-dry fronts [6,7]. Eqs. (1) and (2) become

$$\frac{\partial}{\partial t}h+\frac{\partial}{\partial x}q=0 \tag{3}$$

$$\frac{\partial}{\partial t}q+\frac{g}{2}\frac{\partial}{\partial x}h^2=gh(S-J) \tag{4}$$

1.2. SVE analytical solutions

Analytical solutions are sought mainly for their ability to attest the convergence and correctness of numerical models when a full analytical solution for the problem does not exit. A brief historical

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review of SVE Dam-Break characteristic based analytical solution is presented herein.

SVE Dam-Break analytical solutions are among the most sought solutions. One of the first solutions was presented by Barré de Saint Venant [2] and Ritter [8] for the Dam-Break problem with a dry front. This solution is a parabola that describes the depth of water after the sudden complete breaking of the dam and is based upon the assumption of a prismatic channel with horizontal bed, infinite length and no bed friction. The initial conditions are of a predefined depth upstream of the dam site and no water downstream. The breaking of the dam is assumed to be total and instantaneous. This situation leads to a horizontal asymptote making the propagation in the tip very fast. Dressler [9], by transforming the equations into diffusive wave equations, or, Whitham [10], by treating the tip as a "boundary layer", proposed that the tip of the front wave had a different configuration other than the asymptote. Ritter [8] solution was used and only the tip was changed. Stoker [11] presented the solution for the Dam-Break for a non-wet front where a rarefaction wave travels backwards and a shock wave or bore travels forward. Stoker's solution also incorporated Ritter [8]'s solution if the depth downstream was assumed to be equal to 0. Hunt [12–14] proposed an approximate solution based on the kinematic wave for an infinite wet prismatic channel with slope [12], for a sloped prismatic channel with variable width [14] and for an infinite sloped prismatic channel by using the method of asymptotic expansions. Hunt's work mainly focused on the long waves and is only valid after the wave travelled some distance downstream. More recently Mangeney et al. [15] derived the solution for the 1D sloped Dam-Break with friction using the Method of Characteristics. Ancey et al. [16] presented a solution for steep slopes.

The aim of this work is to: (a) present two analytical solution, for the Dam-break problem based on LInE by using the Method of characteristics and a steady state solution and (b) compare the analytical solutions with two numerical solutions of first and second order, with and without shock capturing ability. In the Methodology section, the formulae for the depth and velocity will be derived and explained along with the wave propagation characteristics for the Dam-Break. Well established numerical schemes will be applied to the LInE, and compared to the analytical solutions. In the last section conclusions will be drawn about the propagation, analytical solutions and numerical schemes.

2. Methodology

2.1. Analytical Dam-Break solution

The initial conditions are constant with a single jump discontinuity at some point [17] usually $x=0$ m and described by

$$h(x,0)=\begin{cases} h_l & \text{if } x<0 \\ h_r & \text{if } x>0 \end{cases} \tag{5}$$

The solution of the Dam-Break for the hyperbolic non-linear LInE (Eqs. (3) and (4)) is obtained through the Method of Characteristics (MOC) that is derived from the geometric theory of the quasi-linear differential equations. MOC provides an insight into the physical behavior and the construction of an analytical solution [18]. In order to derive the analytical solution of the Dam-Break for the LInE two major steps are defined: calculation of (1) the Riemann Invariants and the characteristics and (2) the depth and velocity for the entire domain.

2.1.1. LInE characteristics and Riemann invariants

The concept of Riemann Invariants and characteristics is of the utmost importance to understand the propagation of the waves in a set of hyperbolic equation. Eqs. (6) and (7) are the conservative form of the homogeneous LInE without the source terms. These are valid for a rectangular, horizontal, and with constant width unitary channel.

$$\frac{\partial}{\partial t}h+\frac{\partial}{\partial x}uh=0 \tag{6}$$

$$\frac{\partial}{\partial t}uh+\frac{g}{2}\frac{\partial}{\partial x}h^2=0 \tag{7}$$

To obtain the Riemann Invariants and the characteristics it is first necessary to linearize the previous set of equations and transform them into a celerity–velocity formulation [18]. From Eqs. (6) and (7) by applying the chain rule we obtain:

$$\frac{\partial}{\partial t}h+h\frac{\partial}{\partial x}u+u\frac{\partial}{\partial x}h=0 \tag{8}$$

$$h\frac{\partial}{\partial t}u+u\frac{\partial}{\partial t}h+gh\frac{\partial}{\partial x}h=0 \tag{9}$$

With the celerity $c=\sqrt{gh}$, differentiating c in time and space one obtains

$$\frac{\partial}{\partial t}c=\frac{\partial}{\partial t}\sqrt{gh}=\frac{g}{2\sqrt{gh}}\frac{\partial}{\partial t}h=\frac{g}{2c}\frac{\partial}{\partial t}h \Longrightarrow \frac{\partial}{\partial t}h=\frac{2c}{g}\frac{\partial}{\partial t}c \tag{10}$$

$$\frac{\partial}{\partial x}c=\frac{\partial}{\partial x}\sqrt{gh}=\frac{g}{2\sqrt{gh}}\frac{\partial}{\partial x}h=\frac{g}{2c}\frac{\partial}{\partial x}h \Longrightarrow \frac{\partial}{\partial x}h=\frac{2c}{g}\frac{\partial}{\partial x}c \tag{11}$$

Multiplying (8) by cg and (9) by g and introducing (10) and (11) into (8) and (9), adding and subtracting the equations one obtains

$$\frac{\partial}{\partial t}(2c^3+3uc^2)+c\frac{\partial}{\partial x}(2c^3+3uc^2)=0 \tag{12}$$

$$\frac{\partial}{\partial t}(2c^3-3uc^2)-c\frac{\partial}{\partial x}(2c^3-3uc^2)=0 \tag{13}$$

These equations have the form:

$$\frac{\partial}{\partial t}R+\frac{dx}{dt}\frac{\partial}{\partial x}R=0 \tag{14}$$

With $R=2c^3+3uc^2$ or $R=2c^3-3uc^2$, since $\frac{\partial}{\partial t}R=0$ along the curves represented by the equation $\frac{dx}{dt}=c$ or $\frac{dx}{dt}=-c$ one obtains

$$\frac{\partial}{\partial t}(2c^3+3uc^2)=0 \tag{15}$$

On the positive characteristic curves (C^+) with equation:

$$\frac{dx}{dt}=c \tag{16}$$

And

$$\frac{\partial}{\partial t}(2c^3-3uc^2)=0 \tag{17}$$

On the negative characteristic curves (C^-) with equation:

$$\frac{dx}{dt}=-c \tag{18}$$

On the curves C^+ and C^- the values $2c^3+3uc^2$ and $2c^3-3uc^2$ are the respective Riemann invariants.

2.1.2. Dam-Break

In order to derive the analytical solution for the LInE Dam-Break – following Stoker [11] – one has to divide the structure of the generic fully developed Dam-Break ($t=t_0$) into 4 areas (Fig. 1).

- Area 0 is the downstream condition $depth=h_0$ and $velocity=u_0=0$, limited upstream by the steep front wave, which travels with a constant speed ξ.
- Area 1 is upstream condition and has the initial conditions $depth=h_1$ and $velocity=u_1=0$. These areas are also the initial condition to the Riemann Problem.