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Geometry of finite deformations and time-incremental analysis

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ABSTRACT

In connection with the origin of computational mechanics and consequent progress of incremental methods, a fundamental problem came up even in solid mechanics – namely how to correctly time-linearize and time-integrate deformation processes within finite deformations. Contrary to small deformations (actually infinitesimal), which represent a correction of an initial configuration in terms of tensor fields and so a description by means of a linear vector space of all symmetric matrices $sym(3,\mathbb{R})$ is well-fitting, a situation with finite deformations is rather more complicated. In fact, while the position and shape of a deformed body take place in the usual three-dimensional Euclidean space $\mathbb{R}^3$, a corresponding progress of deformation tensor makes up a trajectory in $Sym^+(3,\mathbb{R})$ – a negatively curved Riemannian symmetric manifold. Since this space is not a linear vector space, we cannot simply employ tools from the theory of small deformations, but in order to analyze deformation processes correctly, we have to resort to the corresponding tools from the differential geometry and Lie group theory which are capable of handling the very geometric nature of this space. The paper first briefly recalls a common approach to solid mechanics and then its formulation as a simple Lagrangian system with configuration space $Sym^+(3,\mathbb{R})$. After a detailed exposition of the geometry of the configuration space, we finally sum up its consequences for the time-incremental analysis, resulting in clear and unambiguous conclusions.

1. Introduction

Significant progress in the finite deformation theory permanently continues, but particularly in recognizing the space of all symmetric positive-definite matrices $Sym^+ \equiv Sym^+(3,\mathbb{R})$ as the configuration space for finite deformations in formulation of solid mechanics as a simple Lagrangian system, it is only now that the full picture of the finite deformations is emerging with its clarity.

The paper serves as an extension to papers [14,15], in that it discusses in detail the geometry of the configuration space for finite deformations Sym^+ – a negatively curved Riemannian (globally) symmetric manifold from its intrinsic geometric point of view. The key idea that the space of all deformation tensors of a body does have naturally defined on it a simple geometric structure – a Riemannian metric that plays a very interesting and crucial role in the development of the theory, has been so far overlooked. However, only utilization of this fact results in the geometrically based approach to solid mechanics via simple Lagrangian system on the space of symmetric positive-definite matrices (deformation tensors), which offers new approach to old problems. This approach is especially appealing, since it enables to utilize the tools of differential geometry and Lie group theory for the analysis of finite deformation processes to provide geometrically justified, unambiguous answers to the time-linearization [14], as well as to its inverse – the time-discrete integration [15].

The Riemannian geometry naturally enters solid mechanics via deformation tensor fields. All four possible deformation tensors, their time derivatives, as well as their corresponding conjugate stresses will be introduced in Section 2. This section serves as a preliminary item with the purpose to provide a concise summary of all the quantities in continuum mechanics, which will be introduced as the geometric (i.e. coordinate-independent) objects within the framework of Riemannian geometry.

Section 3 is the central part of the paper. Its Subections 3.1 and 3.2 are devoted to a detailed exposition, different from that in [14], of Riemannian geometry of the configuration space Sym^+ – the space of deformation tensors. Since finite deformation processes can be thought of as trajectories in the space of all symmetric positive-definite matrices Sym^+, solid mechanics can be treated as a simple Lagrangian system with the configuration space Sym^+, plus the Riemannian metric on it that couples stresses and deformation rates into the stress power. Identification of Riemannian metric on Sym^+ in Section 3.3 then makes it possible to accomplish a geometrically consistent time-linearization of deformation processes, whereas Lie group transformations in Section 3.5 and treating Sym^+ as a homogeneous space in Section 3.6 permit their time-discrete integration.

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In particular, tools of Riemannian differential geometry in Section 4 enable to construct the stress rate in terms of the covariant derivative geometrically and so unambiguously – the result is the Zaremba–Jaumann time derivative. They also make it possible to reveal the geometric meaning of the logarithmic strain, allowing its generalization for strained initial configurations. In Section 5, tools of Lie group theory enable to recognize a well-known relation of solid mechanics as an evolution equation of Lie-type for finite deformations and then suggest a geometrically consistent time-discrete integration of deformation processes, which corrects established numerical schemes. Mathematical preliminaries are mainly summarized in Appendix.

We envision that this exposition will be useful to all researchers in such front-line areas of finite deformations that draw on differential geometry. In Section 2, we aimed at making available a handy reference of basic quantities of solid mechanics introduced geometrically, which may be helpful to readers who want to learn the field, or various aspects of it. At the same time, in Section 3–5, we attempted to write this exposition so that it covers some novel ideas and facts for finite deformations that cannot be found elsewhere, except in papers meant for specialists in various other areas.

2. Riemannian geometry of $\mathbb{R}^3$– quantities of solid mechanics as geometric objects

In this section we shall briefly resume geometric construction of all the quantities of continuum mechanics and identify them with geometric (i.e. coordinate-independent) objects, see [19,27,30,31,52,59–61,65] and cf. [46–51]. Since the lack of a coordinated terminology and notation is a formidable barrier, we chose to keep to nomenclature adopted in [31], so as not to cloud the picture with the many complications and much opacity that a treatment of finite deformations in terms of uncommon terminology and notation would bring.

2.1. Deformation

Deformation of a body is described *globally* in terms of an injective differentiable mapping (diffeomorphism) $\Phi : B \subset \mathbb{E}^3 \to \mathbb{E}^3$, where $\mathbb{E}^3$ stands for the usual three-dimensional Euclidean point space $\mathbb{R}^3$ considered now as a *Riemannian manifold*. Unlike fluid mechanics, which is usually described in terms of velocity fields – time-linearized diffeomorphisms, for solid mechanics there is a distinguished natural referential system, and so its *local* description is described in terms of deformation tensors – actually Riemannian metrics based on space-linearized diffeomorphisms.

The mapping $\Phi : B \to S$ between two manifolds induces the linearized *tangent mapping* $T\Phi$ between the corresponding tangent spaces $T_X\Phi : T_XB \to T_xS$, where $x = \Phi(X)$. Here, B stands for referential and $S = \Phi(B)$ for actual configurations. In solid mechanics, the tangent mapping is usually denoted as $\mathbf{F}{:=}T\Phi$, and called "deformation gradient". To the tangent mapping $T\Phi$, one can assign its *dual* $(T_X\Phi)^* : T_x^*S \to T_X^*B$, and its *transposed* $(T_X\Phi)^{\mathrm{T}} : T_xS \to T_XB$ mappings (see Remark 3). The tangent with its dual mapping can then be extended to define *push-forward* Φ_* and *pull-back* Φ^* operations between corresponding spaces of tensors of any order. For more see Remark 4 in Appendix.

Locally, deformation is characterized by deformation tensor fields – most frequently by the field of the *right Cauchy-Green deformation tensors* $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, but equivalently also by deformation fields of the *left Cauchy-Green* $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, *Piola* $\mathbf{B} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}$, or *Almansi* $\mathbf{c} = \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}$ *tensors*. These deformation fields are actually Riemannian metrics that describe the geometry in the reference (resp actual) picture, obtained by the pull-back (resp push-forward) of the actual (resp reference) picture.

In fact, let us denote by g a Riemannian metric on $\mathbb{E}^3$, whose value at each point $x \in \mathbb{E}^3$ determines a scalar product of any two vectors emanating from this point, and so establishes a geometry in its vicinity. By G we fix a metric on reference configuration. The key notion – a Riemannian metric is a smooth symmetric positive-definite covariant tensor field of second order that determines local geometry in the vicinity of any point (see Appendix). As for nomenclature, 2-tensors will be labeled in italic, but their specific representation as linear mappings in bold. Covariant 2-tensors will be denoted by $\flat$, contravariant by $\sharp$, and mixed without superscript (except for Riemannian metrics g and G, which are covariant by definition), cf. Remarks 1 and 2.

Now, if we express the transposed deformation gradient as $\mathbf{F}^{\mathrm{T}} = \mathbf{G}^{-1}\mathbf{F}^*\mathbf{g}$ by (A.16), then for the mixed *right Cauchy-Green deformation tensor* field we get

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{G}^{-1}\mathbf{F}^*\mathbf{g}\,\mathbf{F} = \mathbf{G}^{-1}\boldsymbol{\Phi}^*(\mathbf{g}), \tag{1}$$

where $\boldsymbol{\Phi}^*(\mathbf{g})$ denotes the *pull-back* transformation of metric $\mathbf{g}$ from actual to referential configuration, cf. (A.26). Since $\mathbf{C}^\flat = \mathbf{G}\mathbf{C}$ by (A.18), we can rewrite (1) into its simpler *covariant form* $\mathbf{C}^\flat = \boldsymbol{\Phi}^*(\mathbf{g})$, that is $C^\flat = \Phi^*(g)$ provided we abandon specific representation of Remark 2. We thus conclude that *the covariant* RIGHT CAUCHY-GREEN *deformation tensor field* $C^\flat = \Phi^*(g)$ *is a Riemannian metric on B*, and because of

$$C^\flat(U, V) \equiv \langle \mathbf{C}^\flat U, V\rangle_{T_XB} \equiv G(\mathbf{C}U, V) = g(\mathbf{F}U, \mathbf{F}V) \equiv g(u, v) \tag{2}$$

due to (A.20), *it describes the local geometry of the deformed body S from the point of view of an observer attached to the undeformed body*. For more about the scalar product of (2), see (A.8) and (A.10) in Appendix. Here, vectors before deformation $U, V \in T_XB$, which transform into $u, v \in T_xS$ after deformation, are according to (A.23) interrelated by $u = \Phi_*(U) \equiv \mathbf{F}U\circ\Phi^{-1}$ and $v = \Phi_*(V) \equiv \mathbf{F}V\circ\Phi^{-1}$.

Similarly, for the mixed *Piola deformation tensor* field we have

$$\mathbf{B} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} = \mathbf{F}^{-1}\mathbf{g}^{-1}\mathbf{F}^{-*}\mathbf{G} = \boldsymbol{\Phi}^*(\mathbf{g}^{-1})\mathbf{G}, \tag{3}$$

where again $\boldsymbol{\Phi}^*(\mathbf{g}^{-1})$ stands for the pull-back from actual to referential configuration of metric in dual space of covectors $\mathbf{g}^{-1} \equiv \mathbf{g}^\sharp$, cf. (A.11) and (A.26). Since $\mathbf{B}^\sharp = \mathbf{B}\mathbf{G}^{-1}$ by (A.18), relation (3) can be rewritten into *contravariant form* $\mathbf{B}^\sharp = \boldsymbol{\Phi}^*(\mathbf{g}^{-1})$, that is $B^\sharp = \Phi^*(g^\sharp)$. Now, we can express the scalar product of two covectors in actual configuration in terms of the Piola tensor. Since covectors before deformation $A, D \in T_X^*B$ transform after deformation into covectors $a, d \in T_x^*S$, they are interrelated by $a = \Phi_*(A) = \mathbf{F}^{-*}A\circ\Phi^{-1}$ and $d = \Phi_*(D) = \mathbf{F}^{-*}D\circ\Phi^{-1}$, cf. (A.24). Then due to (A.5)

$$B^\sharp(A, D) \equiv \langle A, \mathbf{B}^\sharp D\rangle_{T_XB} = \langle \mathbf{F}^{-*}A, \mathbf{g}^{-1}\mathbf{F}^{-*}D\rangle_{T_xS} \equiv \langle a, \mathbf{g}^{-1}d\rangle_{T_xS} \equiv g^\sharp(a, d). \tag{4}$$

The PIOLA *deformation field* $B^\sharp = \Phi^*(g^\sharp)$ *thus describes the local geometry of the deformed body from the viewpoint of an observer attached to the reference configuration, now in terms of covectors.*

We can also proceed reversely and characterize the deformation by the local geometry of the body B from the viewpoint of an observer attached to the actual configuration S. In fact, making use of the *push-forward* transformation $\boldsymbol{\Phi}_*$, cf. (A.27), we get the *left Cauchy-Green deformation tensor* field $b^\sharp = \Phi_*(G^\sharp)$

$$\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{G}^{-1}\mathbf{F}^*\mathbf{g} = \Phi_*(\mathbf{G}^{-1})\mathbf{g}, \tag{5}$$

or the *Almansi deformation tensor* field $c^\flat = \Phi_*(G)$

$$\mathbf{c} = \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1} = \mathbf{g}^{-1}\mathbf{F}^{-*}\mathbf{G}\mathbf{F}^{-1} = \mathbf{g}^{-1}\Phi_*(\mathbf{G}). \tag{6}$$

Expressions for corresponding scalar products in reference configuration, expressed in actual configuration, are analogous to