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Computing the structural buckling limit load by using dynamic relaxation method

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ABSTRACT

The numerical structural analysis schemes are extensively developed by progress of modern computer processing power. One of these approximate approaches is called "dynamic relaxation (DR) method." This technique explicitly solves the simultaneous system of equations. For analyzing the static structures, the DR strategy transfers the governing equations to the dynamic space. By adding the fictitious damping and mass to the static equilibrium equations, the corresponding artificial dynamic system is achieved. The static equilibrium path is required in order to investigate the structural stability behavior. This path shows the relationship between the loads and the displacements. In this way, the critical points and buckling loads of the non-linear structures can be obtained. The corresponding load to the first limit point is known as buckling limit load. For estimating the buckling load, the variable load factor is used in the DR process. A new procedure for finding the load factor is presented by imposing the work increment of the external forces to zero. The proposed formula only requires the fictitious parameters of the DR scheme. To prove the efficiency and robustness of the suggested algorithm, various geometric non-linear analyses are performed. The obtained results demonstrate that the new method can successfully estimate the buckling limit load of structures.

1. Introduction

The structural stability can be evaluated by estimating its critical load. In general, an eigenvalue problem should be solved to calculate this load [1]. In this approach, the aforesaid load makes the stiffness matrix singular, in which the related determinant is zero. It is worth emphasizing that this technique is unsuitable for large structures since it consumes a great deal of time [2]. Alternatively, it is possible to use the equilibrium path for computing the critical load. In this strategy, the critical load is the maximum load before reaching the first limit load or limit displacement points [3–6]. Based on this, three types of load–displacement curves are resulted from the structural responses. In the first one, no buckling occurs, whereas the second one, in which no bifurcation point emerges, has the limit buckling point. The third curve includes the bifurcation point in addition to the limit point. In the last case, the buckling may occur before or after the bifurcation points [5]. It can be emphasized that the bifurcation points generally appear in structures with initial defects [6]. So far, various solution techniques have been used for post-buckling analysis of structures. An improved arc-length method with high-efficiency was proposed by Ju-fen and Xiao-ting [7]. They utilized this procedure to analyze the composite structures. Lee et al. applied FSDT and the von-Kármán non-linear strain–displacement relationship for post-buckling analysis of the FGM plates under edge compression and thermal loads [8]. Asemi and Shariyat investigated effects of the materials auxetic on uniaxial and biaxial post-buckling behaviors of the FGM plates [9]. Uniaxial and biaxial post-buckling behaviors of the longitudinally graded plates were studied by Shariyat and Asemi [10].

The DR method is an approximate numerical procedure. For the first time, this scheme was introduced by Day in 1965 [11]. In this algorithm, a static problem is converted to a fictitious dynamic one. In other words, the mass and damping are added to the static relations for forming an artificial dynamic system [12]. This method is based on the second-order Richardson's approach which is developed by Frankel [13]. Then, Rushton applied this technique to non-linear problems. This researcher used the DR strategy to perform the geometric non-linear analysis of bending plates [14]. It should be noted that the mass and damping matrices play important roles in preserving the stability of the DR process. Furthermore, the time step has also influenced the convergence rate. Researchers presented various procedures for estimating the fictitious parameters of the DR scheme.

In this paper, the DR process is employed in tracing the structural equilibrium path and computing the buckling load. At the

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first stage, the formulas of the DR procedure are presented. Afterwards, the best schemes for calculating the fictitious parameters are reviewed. Then, a new approach is suggested for finding the load factor. To achieve this goal, the work increment of external forces is written in terms of the DR artificial parameters. By setting this increment to zero, a new formula is obtained for determining the load factor. It should be reminded that the work increment is dependent on the displacement and load factor increments. Moreover, the displacement increment is related to velocity. Furthermore, velocity depends on the fictitious mass and damping. As a result, the proposed relation is only associated with the parameters of the DR scheme. This factor is employed in estimating the structural buckling limit load. It should be noted that the main aim of this paper is the calculation of buckling limit load. This load is estimated by the equilibrium path of structures. In other words, at first, the equilibrium path of structures is plotted. Based on this path, the corresponding load to the first limit point is known as buckling limit load.

2. The dynamic relaxation method

The basic relations of the DR technique are achieved by using the central finite difference formulation. These equations are presented in the below form:

$$\dot{X}_i^{n+\frac{1}{2}} = \frac{2m_{ii}^n - C_{ii}^n t^n}{2m_{ii}^n + C_{ii}^n t^n}\dot{X}_i^{n-\frac{1}{2}} + \frac{2t^n}{2m_{ii}^n + C_{ii}^n t^n} r_i^n, r_i^n = p_i^n - f_i^n, i = 1, 2, ..., \text{ndof} \quad (1)$$

$$X_i^{n+1} = X_i^n + t^{n+1}\dot{X}_i^{n+\frac{1}{2}},\ i = 1, 2, ..., ndof \quad (2)$$

In these relations, the i-th entry of the fictitious mass and damping matrices, the artificial time step, the i-th entry of the internal and residual force vectors are m_{ii}^n,C_{ii}^n,t^n f_i^n and r_i^n, correspondingly. It should be noted that superscript n denotes the n-th iteration of the DR strategy. Additionally, the external force of the static system and the number of degrees of freedom are indicated by p_i^n and $ndof$, respectively. Moreover, the displacement and velocity vectors are shown by X and $\dot{X}$, correspondingly. Noticeably, the DR process is explicit. In other words, only vector operators are required in the solution process. This characteristic is the result of the diagonal mass and damping matrices. It should be highlighted that Eqs. (1) and (2) are successively used to achieve an acceptable error. To estimate the fictitious parameters, various procedures are suggested.

Brew and Brotton suggested a method in which the mass of each degree of freedom was assumed to be proportional to its diagonal element in the structural stiffness matrix [15]. Bunce computed the critical damping by utilizing the Rayleigh's principle [16]. By deploying Gershgorin's circle theory, Cassell and Hobbs calculated the fictitious mass [17]. Papadrakakis presumed that both mass and damping are proportional to the stiffness matrix diagonal entries [18]. Underwood suggested a well-known formulation for iterations of the DR method [12]. Besides, by using the Rayleigh's principle, the damping factor and the time step are calculated by Qiang [19]. Zhang and Yu proposed a new formula for estimating the artificial damping, which is based on the Rayleigh's principle [20]. Zhang et al. used nodal damping, in which the degrees of freedom for each node have similar damping [21]. Rezaiee-Pajand et al. and Kadkhodayan et al. minimized the residual force and proposed a new relation for calculating the time step [22,23].

Another well-known formulation is called the kinetic DR method. In this scheme, the damping is ignored. In other words, the damping matrix is set zero in Eq. (1) [24]. Rezaiee-Pajand and Alamatian minimized the displacement errors of the two successive iterative steps. In this way, new formulas were presented for calculating the artificial mass and damping [25]. Rezaiee-Pajand and Sarafrazi utilized the power iteration process for finding the minimum eigenvalue [26]. By minimizing the error between two successive iterations, Rezaiee-Pajand et al. proposed another approach for computing the damping matrix [27]. In another study, Rezaiee-Pajand and Sarafrazi set the damping parameter to zero, and proposed a formula for the time step ratio [28]. Alamatian recommended a new formula for estimating the fictitious mass of the kinetic DR technique [29]. Rezaiee-Pajand et al. found a new time step by minimizing the unbalanced energy function of the fictitious dynamic system, [30]. Finally, Rezaiee-Pajand and Rezaiee presented a new time step for the kinetic DR strategy [31].

In most of the DR approaches, the artificial damping is estimated by the Rayleigh's principle [20]. This relationship is demonstrated as follows:

$$\mathbf{C} = 2\mathbf{M}\sqrt{\frac{\mathbf{X}^T\mathbf{F}}{\mathbf{X}^T\mathbf{M}\mathbf{X}}} \quad (3)$$

In this relation, the stiffness matrix and the internal force vector are shown by **S** and **F**, respectively. The absolute values of the stiffness matrix entries in a row are calculated and summed to find the mass of corresponding degree of freedom, as follows [12]:

$$m_{ii} = \frac{t^2}{4}\sum_{j=1}^{ndof}|S_{ij}| \quad (4)$$

3. The proposed formulation

Non-linear behavior of structures has various characteristics corresponding to load and displacement limit points in stable and unstable paths, buckling points and post-buckling behavior. Sometimes, the increase in the load and displacement simultaneously occurs in a stable path. Besides, the reduction in the load value and increase in the displacement may be observed in an unstable path. The existence of limit points in the structural behavior path leads to the analysis complexity. One application of the DR algorithm is tracing the structural equilibrium paths, and consequently estimating the buckling limit load. In common formulations of the DR method, the external load is constant in each loading increment. Therefore, this technique cannot compute the critical load of structures. Nevertheless, several automatic approaches for removing this difficulty have been proposed [3,32–34]. In these strategies, variable external force is deployed in the DR process. The load factor is usually utilized for changing the load level. If the load factor is shown by λ, the following residual force can be obtained:

$$\mathbf{R} = \lambda\mathbf{P} - \mathbf{F} \quad (5)$$

In the last relation, the residual force and reference load vectors are shown by **R** and **P**, respectively. The reference load is also the external force. Fig. 1 illustrates the equilibrium path of a structure with a snap-through point. Firstly, the load and displacement increase simultaneously on the stable path. This increase proceeds until reaching the first limit point. In Fig. 1, a load limit point exists. Afterwards, the load decreases whereas the displacement increases. After limit load, the structure becomes unstable [6]. The part of this equilibrium path before the limit load is called the pre-buckling region, and the section after this point is the post-buckling part. It is worth emphasizing that the load before the first load or displacement limit point denotes the buckling limit load of the structure [3–6]. This load is shown in Fig. 1 by P_{cr} [4,35].