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Ideal and physical barrier problems for non-linear systems driven by normal and Poissonian white noise via path integral method

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ABSTRACT

In this paper, the probability density evolution of Markov processes is analyzed for a class of barrier problems specified in terms of certain boundary conditions. The standard case of computing the probability density of the response is associated with natural boundary conditions, and the first passage problem is associated with absorbing boundaries. In contrast, herein we consider the more general case of partially reflecting boundaries and the effect of these boundaries on the probability density of the response. In fact, both standard cases can be considered special cases of the general problem. We provide solutions by means of the path integral method for half- and single-degree-of-freedom systems for both normal and Poissonian white noise. Emphasis is put on the considerations of the yielding barrier which is expressed in terms of non-reflecting (but not absorbing) boundary conditions. Comparison with Monte Carlo simulation demonstrates the excellent accuracy of the proposed method.

1. Introduction

The problem of the probability distribution of a random process in the presence of barriers is crucial in structural engineering. The barriers encountered are mainly of two different kinds which we propose to distinguish as follows: *ideal* and *physical*. The former is a barrier that does not modify the equation of motion and it is useful for reliability analysis. The latter situation occurs when at least one state variable is constrained between one (or two) walls, or when the parameters of the system instantaneously change as some threshold is reached.

The ideal barrier may be *absorbing* or not. The absorbing barrier acts in such a way that when a state variable reaches it for the first time the trajectory is cancelled. This means that the probability content of the surviving trajectories will decrease monotonically. This is the so-called *first passage problem* [1–9]. Finding the PDF of the first passage time is crucial for safety analysis.

If a state variable touches the barrier and the corresponding trajectory survives, then the barrier is not absorbing. This kind of barrier is useful for the level-crossings of a random process, which in turn is expressed by so-called *counting processes*. In the special case when the barrier ξ is zero, we have the so-called *zero crossing problem*.

The physical barrier is the reflecting one. It occurs when the motion of the system is constrained between one (or two) walls. As an example, a mass may move freely between $-\infty$ and ξ (where ξ is the barrier level), but as it touches the barrier the mass is reflected like it happens during an impact [10–13]. If some quantity of energy is dissipated during the impact (inelastic impact) then the barrier reflects the mass with a restitution factor c_r with $0 \le c_r \le 1$. The two extremes are $c_r = 0$ (purely inelastic impact, the mass remains glued to the barrier) and $c_r = 1$ which is elastic impact. This latter case means that the velocity before and after the impact simply changes its sign and the kinetic energy is preserved. The case c_r is related to a moving barrier $\xi(t)$ with the sign of $\dot{\xi}(t)$ opposite to the sign of $\dot{x}(t)$.

Recently, the absorbing barrier problem [14] and the vibroimpact problem [15] have been treated by using the Path Integration (PI) method. The appeal of using such a method for finding the PDF in the presence of strong non-linearities and impacts lies in the fact that the discretized version of the Chapman–Kolmogorov equation (CK), namely the PI, allows us to follow the various trajectories step by step for non-linear systems under normal white noise processes [16,17] and Poissonian white noise [18,19]. It follows that it is possible to control the number of time, in the whole process, that the various trajectories touch or do not touch the barriers. In the case of the ideal barrier we do not modify the equation of motion and we simply count the number of times that the barrier is crossed or we cancel the trajectories crossing the barrier (absorbing barrier). For physical barriers like the ones in

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vibroimpact problems we modify the equation of motion by inserting new initial conditions in the step according to the kind of impact.

In this paper, another kind of barrier is proposed, i.e. the *yielding barrier*. This kind of barrier is useful for finding the probability distribution of elasto-plastic systems. It consists of an impact problem with restitution coefficient $c_r=0$ with the difference that the trajectories remain glued to the barrier until some variable changes its sign. This is exactly what happens to an elasto-plastic oscillator in the stress space. In fact, when the stress attains the yield value s_0, the stress $S(t)$ remains frozen at the barrier as long as $S(t)\dot{S}(t)$ remains non-negative. Otherwise it will move freely in the elastic zone until another impact occurs. Moreover, for completeness all the kinds of barrier problems are analysed with the extension to Poisson white noise. For simplicity, the analysis is first presented for half oscillators, and then extended to single degree of freedom oscillators. The results from PI are compared to Monte Carlo (MC) simulation showing in all cases perfect agreement of the PDF evaluated by PI and that evaluated by MC.

2. Barrier problems

The barrier problems traditionally are of two kinds: *absorbing barrier* and *reflecting barrier*. The former kind is such that if a realisation of the given process touches for the first time a given barrier it disappears (see Fig. 1(a)).

The latter is such that the realisation touches the barrier the trajectory is reflected with the law

$$\dot{X}(t^*)^+ = -c_r\dot{X}(t^*)^- \tag{1}$$

where t^* is the generic time instant at which the impact occurs, the superscripts $^-$ and $^+$ stand for the immediately before and after the impact, respectively, and c_r is the restitution coefficient with $0 \le c_r \le 1$. If $c_r=0$ the impact is inelastic and for all the subsequent time instants at which the trajectory touches for the first time the barrier it remains glued to the barrier (see Fig. 1(b)).

The absorbing barrier problem is useful for the definition of the so-called *reliability function* and for the *first passage* problem [14]. The reflecting barrier is useful to study vibroimpact systems in mechanical and electrical systems. However, as we assume that $c_r = -1$ in Eq. (1) we have that the barrier does not reflect at all and the trajectories touching the barrier will survive. This kind of barrier is useful for the so-called counting process as usually employed for the analysis of level-crossings.

Then we propose to distinguish the barriers as ideal and physical. The former are such that the barrier does not modify neither the parameters of the system nor its initial conditions before and after the impact at time t^*. In contrast to that, a physical barrier will modify either the initial conditions for $t \ge (t^*)^+$ or the parameters of the system.

Among the physical barriers, we propose another one that will be labeled as *yielding barrier*. It consists of a reflecting barrier with restitution coefficient $c_r=0$ with the difference that the trajectory will remain glued to the barrier until another variable (strictly related to the previous one) will change its sign. Then it moves freely inside the barrier unless another impact occurs. This is exactly the behaviour of an elasto-perfectly plastic spring. Such a kind of barrier is a physical one and it is useful for the study of elasto-plastic systems.

A model of such an elasto-plastic system is shown in Fig. 2. It comprises a linear spring (k) in series with a friction element capable of transmitting a stress S bounded by $-s_0 \le S \le s_0$. This spring–friction combination is in parallel to a linear dashpot with viscosity constant c. Its state is uniquely defined by the displacement x and the plastic slip z in the friction element.

The differential equations governing the evolution of the state variables are

$$c\dot{X} + k(X - Z) = F(t) \tag{2}$$

In this equation, $S = k(X-Z)$ denotes the restoring force (or by proper normalization with a reference area which, for convenience, has been set to unity here) the stress. Due to the limitation on the magnitude of the stress, the rate of the plastic slip is determined by the relations

$$\dot{Z} = \begin{cases} \dot{X} & S = s_0 \wedge \dot{X} > 0 \\ \dot{X} & S = -s_0 \wedge \dot{X} < 0 \\ 0 & \text{else} \end{cases} \tag{3}$$

Since $\dot{S} = k(\dot{X} - \dot{Z})$, we can re-formulate the equation of motion in terms of the stress variable using

$$c\dot{X} = c\dot{Z} + \frac{c}{k}\dot{S} \tag{4}$$

and using Eq. (2)

$$\frac{c}{k}\dot{S} + S = F(t) + c\dot{Z} \tag{5}$$

Since for the cases $S = s_0 \wedge \dot{X} > 0$ as well as $S = -s_0 \wedge \dot{X} < 0$ we have $\dot{S} = 0$ and for the remaining cases we have $\dot{Z} = 0$, the above equation upon introducing $\alpha = c/k$ reduces to

$$\alpha\dot{S} = 0; \quad |S| = s_0 \wedge S\dot{S} \ge 0 \tag{6a}$$

$$\alpha\dot{S} + S = F(t); \quad \text{else} \tag{6b}$$

The condition $\dot{S} \ge 0$ ($\dot{S} \le 0$) is equivalent to the condition $\dot{X} \ge 0$ ($\dot{X} \le 0$). It can be seen that the evolution of the stress $S(t)$ can be formulated independently, i.e. without explicit specification of X and Z.

The relations between the displacement Z, the plastic slip Z and the stress S can be seen in Fig. 3.

It can be seen that the stress $S(t)$ remains glued to the yield stress s_0 as long as $S\dot{S}$ remains non-negative, otherwise an elastic recovery occurs and $S(t)$ returns below the barrier moving

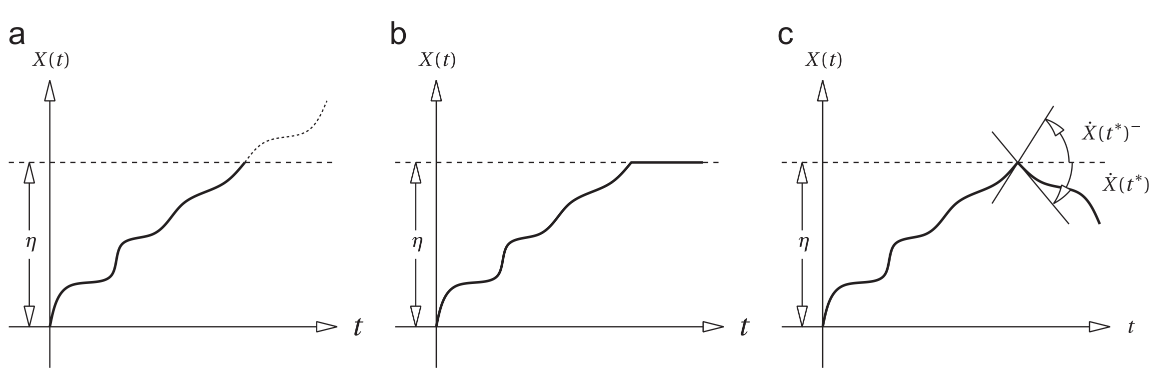

Fig. 1. Classical barriers.