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# An unconstrained mathematical description for conduction heat transfer problems with linear temperature-dependent thermal conductivity

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ABSTRACT

This work presents a systematic modeling for conduction heat transfer problems in which the thermal conductivity is assumed a linear function of the temperature. In general, the mathematical descriptions arising from a linear relationship between thermal conductivity and temperature give rise to more than one solution, some of them without physical sense. In this work a convenient mathematical representation is proposed, avoiding physically inadmissible solutions.

A conduction heat transfer problem in which the thermal conductivity decreases linearly with the temperature in a given interval is considered in this work. A physically equivalent alternative form, valid for any absolute temperature is proposed, giving rise to an unrestricted mathematical modeling and circumventing the need of a posterior choice for establishing the solution with physical meaning.

An equivalent minimum principle for the problem is presented, showing that the extremum of a proposed functional corresponds to the solution of the problem.



## 1. Introduction

One of the most important issues, when describing a nonlinear heat transfer problem, is to avoid a physically inadmissible solution. When carrying out a computational simulation, we must avoid mathematical descriptions which give rise to more than one solution (from a strictly mathematical point of view) but admit, at maximum, one solution with physical sense.

Numerical simulation of a problem represented by equations admitting more than one solution imposes an additional analysis for choosing the solution with physical sense. Sometimes it requires an algorithm change, in order to avoid a solution without physical sense, previously obtained in a simulation.

Anyway, in order to minimize costs and effort, when dealing with a problem that admits at maximum one solution with physical meaning, we must work with an unrestricted mathematical modeling – without inequalities – which admits only the physical solution, provided, of course, that it exists and is unique.

Although the simple mathematical representation leads to treat conduction heat transfer problems supposing constant thermal conductivity, physically realistic problems generally present temperature-dependent thermal conductivity and, in many cases, a strong dependence can be verified, with the thermal conductivity assuming different orders of magnitude, even for a moderate temperature variation. This is particularly important when the thermal conductivity decreases with a temperature increase. Glassbrenner and Slack [4] have measured the thermal conductivity of silicon and germanium, both strongly dependent on the temperature. While at 100 K the thermal conductivity of Silicon is 950 W/mK; at 200 K it is 266 W/mK; at 500 K, 80 W/mK and at 1000 K 31 W/mK. For the germanium, at 100 K the thermal conductivity is 225 W/mK; at 200 K it is 95 W/mK; at 500 K, 33.8 W/mK and at 1000 K, 17.1 W/mK. Aiming at investigating the properties of high-purity isotopically enriched substances, since the isotopic composition of a substance may have a significant effect on the temperature dependence of thermal conductivity, Gusev et al. [5] have measured thermal conductivity of isotopically

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enriched monoisotopic silicon (99.896% purity) in the range from 80 to 300 K.

Yevtushenko and Kulchytsky-Zhyhailo [18] proposed an approximate method to solve the contact problem for two heated bodies with the thermal conductivity strongly decreasing with temperature, since one of the bodies was carbon graphite. Sevostianov and Mishuris [13] addressed homogenization procedures for obtaining effective thermal conductivity of heterogeneous materials composed by isotropic constituents with thermal conductivities proportional to each other and temperature-dependent properties.

Carbon nanotubes present important nanoscale applications like very high thermal conductivity along the axial direction. Only recently, Mehri [7] investigated the role of temperature difference in heat transport, finding out a decrease in the thermal conductivity with increase in the temperature difference imposed on nanotube, although many studies have focused on the dependence of thermal conductivity on temperature. For instance, Osman and Srivastava [9] observed a peaking behavior in the thermal conductivity as a function of temperature, before falling off at higher temperatures; Hone et al [6] observed that the temperature-dependent thermal conductivity of crystalline ropes of single-walled carbon nanotubes decreases smoothly with decreasing temperature, displaying a linear temperature-dependence below 30 K.

The temperature-dependent thermal conductivity was estimated by a direct method proposed by Kim (see [12]), transforming the steady-state non-linear heat conduction equation without heat source into the Laplace equation by employing Kirchhoff transformation. In the methodology, the thermal conductivity is determined from the imposed heat flux and the (measured) temperatures at the boundaries. Kirchhoff transformation has also been employed by Sluzalec and Kleiber [14] for deriving the shape design sensitivity expressions for bodies with steady-state non-linear heat conduction, reducing the non-linear thermal problem to the standard Laplace problem. Tomatis [15] applied the transformation to solve the non-linear heat conduction problem in the classical nuclear fuel rod model, in order to solve adequately the non-linear heat transfer equation in the fuel rods, since the fuel temperature is very important to compute the neutron reactivity. Bonani and Ghione [2] employed Kirchhoff transformation in the thermal analysis of high-power semiconductor devices with temperature-dependent and piecewise inhomogeneous thermal conductivity. A non-linear fin problem in which the thermal conductivity and heat transfer coefficient are power law functions of the temperature is treated by Moitsheki et al. [8], who present exact solutions employing classical Lie symmetry techniques.

Temperature-dependent thermal conductivity has already been considered by many authors, some of them cited before, but Rajagopal and Saccomandi [10] performed the first theoretical study of the pressure-dependent thermal conductivity on non-linear elastic solids, and carefully discussed the hypotheses made by Fourier and the dependence of thermal conductivity not only on temperature, but also on pressure (usually accounted for in gases – and sometimes in liquids), for solids.

Saldanha da Gama et al. [12] proposed an a priori temperature estimate, when the thermal conductivity is temperature dependent, considering a piecewise constant behavior. The estimate allows establishing, without any simulation – exactly or numerically – a value greater than (or equal to) the temperature in any point of the body. This a priori estimate may be useful, for instance, when the main goal is to ensure that a (maximum admissible) temperature will not be reached.

The present work may be considered as an improvement of Saldanha da Gama et al. [12], since there is a linear relationship between the thermal conductivity and the temperature, certainly presenting a physically more realistic behavior. Besides, it is important to emphasize that the mathematical representation proposed in this work – with solution existence and uniqueness ensured – is valid for any absolute temperature, always leading to physically admissible solutions. Since an equivalent minimum principle for the heat transfer problem is proposed, the problem solution is the minimum of a convex and coercive functional.

## 2. Problem motivation

In this work, we focus in conduction heat transfer problems in which the thermal conductivity is a decreasing function of the temperature. Although thermal conductivity can be any positive-valued function of the temperature, a physically inadequate behavior can arise from simulations when a decreasing function of temperature is considered, particularly when the Kirchhoff transformation is employed. In general, such behavior is represented by a linear relationship valid in a given interval, like the one presented below

$$k = k_1 + (k_2 - k_1)\left(\frac{T - T_1}{T_2 - T_1}\right) \quad T_1 \le T \le T_2, \quad k_1 \ge k \ge k_2 \tag{1}$$

in which $k_1$, $k_2$, $T_1$ and $T_2$ are always positive constants, since all temperatures are assumed absolute throughout this work, and in which it is expected temperatures within the interval $(T_1,\ T_2)$.

In order to illustrate the objective of this work, let us consider the infinite solid represented in Fig. 1, bounded by the planes $x = -2L$ and $x = 2L$ [3] with a constant internal heat generation (per unit time and unit volume) $\dot{q}$ for $-L < x < L$ and with no heat generation for $-2L < x < -L$ and $L < x < 2L$, assuming a thermal conductivity given by Eq. (1).

Taking into account the symmetry, assuming steady-state conduction heat transfer, in the one-dimensional geometry presented in Fig. 1, the phenomenon is mathematically described by

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \quad L < x < 2L$$
$$k\frac{dT}{dx} = -\dot{q}L \quad \text{at } x = L \quad (\text{from symmetry}) \tag{2}$$

This problem requires a boundary condition at $x = 2L$: the (known) temperature at $x = 2L$ will be denoted by $T_{2L}$ (and the temperature at $x = L$ by $T_L$). Integrating the first equation in Eq. (2), it comes that the second equation is valid in a domain $L \le x \le 2L$, thus

$$\int_{T_{2L}}^{T_L} k\,dT = -\dot{q}L(L - 2L) \Rightarrow \dot{q}L^2 = k_1(T_L - T_{2L}) + \frac{1}{2}\left(\frac{k_2 - k_1}{T_2 - T_1}\right)\left((T_L - T_1)^2 - (T_{2L} - T_1)^2\right) \tag{3}$$

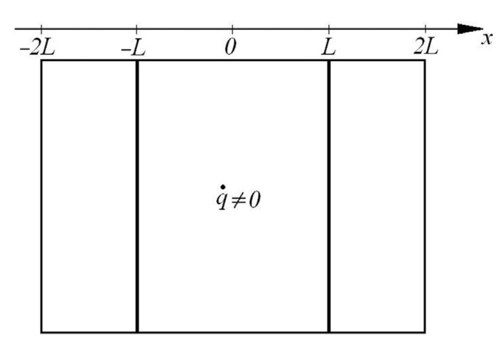


**Fig. 1.** Problem scheme.