International Journal of Non-Linear Mechanics 77 (2015) 1–11

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Non-linear stability and remote unconnected equilibria of shallow arches with asymmetric geometric imperfections

Yang Zhou, Walter Chang, Ilinca Stanciulescu*

Rice University, Department of Civil and Environmental Engineering, Houston, TX 77005, USA

ARTICLE INFO

Article history:
Received 31 March 2015
Received in revised form
19 June 2015
Accepted 22 June 2015
Available online 10 July 2015

Keywords:
Asymmetric geometric imperfections
Remote unconnected equilibria
Nonlinear stability
Shallow arches

ABSTRACT

This paper presents an analytical method to study the non-linear stability and remote unconnected equilibria of shallow arches with non-symmetric geometric imperfections. The exact solutions of the equilibria and critical loads are obtained. Unlike many previous studies, these solutions can be applied to arbitrary shallow arches with arbitrary geometric imperfections. It is found that slightly imperfect arches have multiple remote unconnected equilibria that cannot be obtained in experiments or using finite element simulations if a proper perturbation is not performed. The formulas to directly calculate the critical loads, including those of the remote unconnected equilibria, are also derived. The effect of asymmetric geometric imperfections on the equilibria and critical loads is revealed by applying the derived formulas to half-sine arches with different geometric imperfections.

1. Introduction

Shallow arches are commonly used structural components in civil, aerospace and mechanical engineering. When an arch is subjected to a transverse load, it can display highly non-linear behavior such as snap-through buckling, where the structure loses stability and suddenly jumps to a remote coexisting equilibrium configuration. This process typically involves sudden curvature and stress reversals, which greatly exacerbate the fatigue damage.

Many of the previous studies on non-linear stability of shallow arches have focused on symmetric systems (geometry, boundary and loading conditions). In one of the pioneering studies [1], Fung and Kaplan utilized Fourier series to derive the non-linear equilibrium and buckling equations. Exact solutions of the buckling load were obtained for a sinusoidal arch subjected to a sinusoidal distributed load since only one mode is involved. This early work was extended by several researchers to study the influence of elastic foundations [2,3], to include thermal effects [4], and to derive exact solutions for the sinusoidal arch subjected to a point load [5,6]. Another pioneering work was from Schreyer and Masur [7], where the equilibrium and buckling equations were derived using energy method and solved analytically. Following this work, many studies have been conducted by Pi and Bradford [8–11] and other researchers [12–14], including a variety of cases such as parabolic and circular shapes, concentrated and distributed loads, pinned–pinned, fixed–fixed and elastic boundary constraints. In addition, the non-linear finite element method has also been widely adopted to investigate the non-linear buckling of shallow arches [15–20]. Path following methods are typically used to identify the primary equilibrium path and the limit-point buckling load. To obtain the bifurcated branches, additional numerical techniques are commonly required [21,22].

The arches used in engineering applications usually have certain geometric imperfections, which their buckling and post-buckling behavior can be very sensitive to [23]. The non-linear buckling analysis of geometrically imperfect arches is not without precedence. The early report [1] by Fung and Kaplan also covered a short discussion on half-sine arches with specific geometric imperfections. Maximum three sine terms were adopted to derive the approximate solutions for the case of a concentrated load. The predicted minimum rise for the perfect sinusoidal arch subjected to a concentrated load to display bifurcation buckling was pointed out as *inaccurate* by a recent article [5]. Examples of analytical, asymptotic or numerical solutions for the non-linear buckling of imperfect arches can be found in [24–29]. In all these geometric imperfection analyses, the buckling load was either calculated directly with no knowledge of the postbuckling behavior or by just tracing a single continuous equilibrium branch. Such analyses preclude the exploration of remote unconnected equilibrium paths and their critical states. Pi and Bradford [30] recently demonstrated that it is crucial to identify all equilibrium states including the remote equilibria to determine the branch that the arch can

* Corresponding author. Tel.: +1 713 348 4704; fax: +1 713 348 5268.
E-mail address: ilinca.s@rice.edu (I. Stanciulescu).

http://dx.doi.org/10.1016/j.ijnonlinmec.2015.06.015
0020-7462/

dynamically jump to. Harvey and Virgin [31] experimentally proved the existence of remote unconnected equilibria for shallow arches under unsymmetric loads.

In the present paper, the primary objective is to explore the remote unconnected equilibria and critical loads of shallow arches with asymmetric geometric imperfections. Exact solutions are obtained for equilibrium states and critical loads without truncating the Fourier series. Moreover, these expressions are valid for arbitrary shallow arches with arbitrary geometric imperfections. When these formulas are applied to half-sine arches with different asymmetric geometric imperfections, it is found that many remote unconnected equilibria with multiple limit and bifurcation points exist for these slightly imperfect arches. These remote unconnected equilibria and their critical loads cannot typically be detected in experiments or using the non-linear finite element method by simply varying the control parameter quasi-statically. It is only possible to identify them experimentally or numerically by first performing a perturbation that requires prior knowledge of these unconnected equilibria. In many circumstances, the critical loads of the remote equilibria are much larger than those of the primary equilibrium states for arches with small imperfections. The effect of different geometric imperfections on all critical loads, including those of the remote unconnected equilibria, is investigated thoroughly in this work.

2. Governing equations

In this section, the non-linear equilibrium and buckling equations are derived for pinned–pinned slender shallow arches with arbitrary initial shapes subjected to arbitrary vertical loads f^* (Fig. 1). Both the geometric and load imperfections can be analyzed, but only the asymmetric geometric imperfection is investigated in this paper.

The material is assumed to be elastic, isotropic and homogeneous throughout the entire analysis. To model the geometric non-linearity of slender arches, a large displacement Euler Bernoulli beam theory is adopted. In Fig. 1, E denotes Young's modulus, A and I represent the area and the moment of inertia of the cross section, respectively, and L is the horizontal span of the arch.

2.1. Equilibrium equations

Following [1,5,4], the differential equation, describing the equilibrium states of the shallow arch, can be written as

$$EI(y-y_0)_{,xxxx}-P^*y_{,xx}=f^* \tag{1}$$

where y_0 and y are the initial and deformed shapes of the arch, respectively, the subscript ",x" indicates differentiation with respect to the horizontal position, f^* represents the applied vertical load, and P^* denotes the axial force that can be calculated from the average axial strain over the span:

$$P^*=\frac{EA}{2L}\int_0^L (y_{,x}^2-y_{0,x}^2)\,dx \tag{2}$$

If f^* is a distributed load, it can be written as a load density q^* whose physical dimension is force over length. If f^* is a point load at location x_0, it can then be replaced with $Q^*\delta^*(x-x_0)$, where Q^* is the dimensional load whose dimension is force and δ^* is the dimensional Dirac delta function whose dimension is one over length.

Utilizing $(u,u_0)=(1/r)(y,y_0)$, $\xi=(\pi/L)x$, $p=P^*L^2/\pi^2EI$ and $\delta(\xi)=(\pi/L)\delta^*(x)$, Eqs. (1) and (2) can be non-dimensionalized into the following forms:

$$(u-u_0)_{,\xi\xi\xi\xi}-pu_{,\xi\xi}-q=0 \tag{3}$$

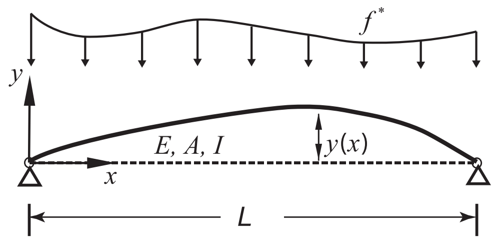

Fig. 1. The shallow arch with an arbitrary initial shape and subjected to an arbitrary transverse load.

$$p=\frac{1}{2\pi}\int_0^\pi (u_{,\xi}^2-u_{0,\xi}^2)\,d\xi \tag{4}$$

where $r=\sqrt{I/A}$ is the radius of gyration of the cross section; $\delta(\xi)$ is a non-dimensional Dirac delta function; $q=q^*L^4/\pi^4EIr$ for the distributed load and $q=Q^*L^3/\pi^3EIr$ for the concentrated load.

Adopting the Fourier sine series that satisfies pin-ended boundary conditions, the initial and deformed configurations, and the non-dimensional external load can be expressed as

$$u_0(\xi)=\sum_{n=1}^{\infty}\beta_n\sin n\xi \tag{5}$$

$$u(\xi)=\sum_{n=1}^{\infty}\alpha_n\sin n\xi \tag{6}$$

$$q=\sum_{n=1}^{\infty}q_n\sin n\xi \tag{7}$$

where

$$q_n=\frac{2}{\pi}\int_0^\pi q\sin n\xi\,d\xi,\quad n=1,2,\ldots \tag{8}$$

Substituting Eqs. (5)–(8) into Eqs. (3) and (4), and equating the coefficients of sin $(n\xi)$ from both sides, the following system of equilibrium equations is obtained:

$$(\alpha_n-\beta_n)n^4+pn^2\alpha_n-q_n=0,\quad n=1,2,\ldots \tag{9}$$

where

$$p=\sum_{k=1}^{\infty}\frac{(\alpha_k^2-\beta_k^2)k^2}{4} \tag{10}$$

2.2. Buckling equations

When the shallow arch loses stability, the tangent stiffness of the system becomes singular. The components of the tangent stiffness matrix of the system can be derived from Eq. (9) as follows:

$$K_{nm}=\frac{\partial R_n}{\partial\alpha_m}=\frac{n^2m^2}{2}\alpha_n\alpha_m+n^2(n^2+p)\delta_{nm},\quad n,m=1,2,\ldots \tag{11}$$

where R_n is the residual (left hand side of Eq. (9)), δ_{nm} is the Kronecker delta, and p is the dimensionless axial force defined in Eq. (10). When the tangent stiffness matrix is singular, its determinant equals zero. Therefore, the buckling equation can be derived as

$$\det\mathbf{K}=\prod_{k=1}^{\infty}\gamma_k+\sum_{n=1}^{\infty}\left(\alpha_n^2\prod_{\substack{k=1\\k\neq n}}^{\infty}\gamma_k\right)=0 \tag{12}$$

where $\gamma_k=2(k^2+p)/k^2$ $(k=1,2,\ldots)$. Since buckling states also need to satisfy the equilibrium condition, Eq. (12) together with (9) and (10) provide the critical loads.