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Traveling waves in one-dimensional non-linear models of strain-limiting viscoelasticity

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ABSTRACT

In this paper we investigate traveling wave solutions of a non-linear differential equation describing the behaviour of one-dimensional viscoelastic medium with implicit constitutive relations. We focus on a subclass of such models known as the strain-limiting models introduced by Rajagopal. To describe the response of viscoelastic solids we assume a non-linear relationship among the linearized strain, the strain rate and the Cauchy stress. We then concentrate on traveling wave solutions that correspond to the heteroclinic connections between the two constant states. We establish conditions for the existence of such solutions, and find those solutions, explicitly, implicitly or numerically, for various forms of the non-linear constitutive relation.

1. Introduction

The present paper is concerned with the dynamics of a viscoelastic medium investigating the traveling wave solutions of the equation

$$T_{xx}+\nu T_{xxt}=g(T)_{tt}, \tag{1.1}$$

where $T(x,t)$ is the Cauchy stress at point x and time t, $g(\cdot)$ is a non-linear function and $\nu>0$ is a constant. Eq. (1.1) is a one-dimensional non-linear differential equation in T resulting from the equation of motion and a constitutive equation relating the stress, the linearized strain and the strain rate.

As opposed to the classical models in mechanics, the strain can be written as a function of the stress, rather than expressing the stress in terms of the kinematical variables. This idea is due to Rajagopal [11,12], who introduced a generalization of the theory of elastic materials by suggesting implicit models allowing for approximations where the linearized strain is a non-linear function of the stress. A series of papers on such implicit theories have been published recently (see e.g. [1,4,5,15,16]). The advantage of this new idea is that it allows for the gradient of the displacement to stay small so that one could treat the linearized strain, even for arbitrary large values of the stress. In this work we focus on four different such models, and we reconsider them in the context of viscoelasticity. We also look at models with quadratically and cubically non-linear constitutive relations although they do not behave as expected for large values of the stress.

There are numerous models introduced by Rajagopal in [11] with implicit constitutive relations between the stress and the strain including models for elastic fluids, inelastic materials and non-hyperelastic materials. Following these models, various forms of non-linear constitutive relations have been studied in different contexts. For example, Kannan et al. [9] worked on the elastic case with a polynomial type non-linearity (see Section 2 for more details). Bulíček et al. [1], on the other hand, considered the static case with a more general non-linearity (see Section 2) and presented the first existence result in a three-dimensional domain.

For viscoelasticity, much less is done in the literature. As explained by Muliana et al. in [10], force, and hence the stress, is the cause for deformation, hence for the strain. Because of this the strain should be described in terms of the stress or its history than vice versa. The motivation for this idea is that in the classical elasticity theory, there cannot be a non-linear relationship between the linearized strain and the stress, which, in fact, is observed in some experiments (see e.g. [19,15]). The fracture of brittle elastic bodies is another possible application area for such implicit theories, where one can obtain bounded strain at the crack tip due to the possibility of having a non-linear relationship between the linearized strain and the stress (see [18] for details). Muliana et al. [10] developed a quasi-linear viscoelastic model where the strain is expressed as an integral of a non-linear measure of the stress. Rajagopal and Srinivasa in [17] proposed a Gibbs-potential-based formulation for the response of viscoelastic materials in this new class. Also Rajagopal and Saccomandi [16] investigated viscoelastic response of solids, a one-dimensional version of which is the one we study in this work, namely

$$\gamma\mathbf{B}+\nu\mathbf{D}=\beta_0\mathbf{I}+\beta_1\mathbf{T}+\beta_2\mathbf{T}^2, \tag{1.2}$$

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where γ and ν are non-negative constants, $\beta_i = \beta_i(I_1, I_2, I_3), (i = 0, 1, 2)$, $I_1 = \text{tr}\,\mathbf{T}, I_2 = \frac{1}{2}\text{tr}\,\mathbf{T}^2, I_3 = \frac{1}{3}\text{tr}\,\mathbf{T}^3$, $\mathbf{B}$ is the left Cauchy–Green stretch tensor and $\mathbf{D}$ is the symmetric part of the gradient of the velocity field. As they explain, this model includes as special subcases; models for a very general new class of elastic and viscoelastic bodies (e.g. Titanium and Gum metal alloys), as well as the Navier–Stokes fluid model (see e. g. [12]). Linearizing the strain in this model reduces (1.2) to

$$\boldsymbol{\epsilon} + \nu \boldsymbol{\epsilon}_t = \beta_0 \mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2, \tag{1.3}$$

where $\boldsymbol{\epsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the linearized strain and $\boldsymbol{u}(\boldsymbol{x}, t)$ is the displacement.

We study (1.3) in one space dimension with a general non-linear right-hand side (see (2.2)). We are interested in analyzing the conditions on the non-linearity $g(T)$ when traveling wave solutions of the form $T(\xi)$ with $\xi = x - ct$, where c represents the wave propagation speed, exist for two constant equilibrium states at infinity. We find the solutions analytically (implicitly or explicitly) or numerically. More precisely, we will first look at the quadratic and the cubic cases for which we are able to solve the problem analytically and obtain explicit or implicit solutions. After that we will study four non-linear models, namely Models A–D (see Section 2), and we will either express the solution implicitly, or obtain it numerically if it is not possible to find an analytical solution. Our work seems to be the first such treatment in the literature of strain-limiting viscoelasticity.

The propagation of traveling waves in non-linear viscoelastic solids has also been studied previously in the context of classical theory of viscoelasticity (see e.g. [6–8], and references therein). The results of present work exhibit some similarities with those in the literature. The first common point is that the equations of motion admit kink-type traveling wave solutions. Also, in both cases, the effective width of the traveling wave is proportional to the viscosity parameter and the wave profile becomes smoother as the viscosity parameter increases. However, our study differs from the articles within the context of classical viscoelasticity theory in the sense that the governing equation in our model (see (1.1)) is in terms of the stress and also the non-linearity is on the inertia term.

The structure of the paper is as follows. In Section 2 we introduce the one-dimensional strain-limiting viscoelasticity model as well as give a list of four non-linear constitutive relations that has been suggested for elastic solids. In Section 3 we consider traveling wave solutions of the governing equations. In Section 4 we solve the resulting differential equation for different non-linear constitutive relations, and give analytical solutions where possible, or obtain numerical solutions.

2. One-dimensional strain-limiting viscoelasticity

Consider a one-dimensional, homogeneous, viscoelastic, infinite medium exhibiting small strains for large stresses. In the absence of external body forces, the equation of motion is given by

$$\rho_0 u_{tt} = T_x, \tag{2.1}$$

where ρ_0 is the mass density of the medium, the scalar-valued function $u(x,t)$ is the displacement, and $T(x,t)$ is the Cauchy stress. Here and throughout this work the subscripts denote partial derivatives. In contrast to explicit constitutive relations of the classical theories of viscoelasticity, we shall employ an implicit constitutive relation

$$\epsilon + \nu \epsilon_t = g(T), \tag{2.2}$$

which gives the linearized strain $\epsilon = u_x$ and the strain rate ϵ_t as a non-linear function of the stress T, with $g(0) = 0$ and a non-negative constant ν. The model defined by (2.2) is the one dimensional form of (1.3). When $\nu = 0$, it reduces to the one-dimensional version of the model introduced by Rajagopal in [11,12] for elastic solids.

For convenience, we now define the dimensionless quantities

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{L}\sqrt{\frac{\mu}{\rho}}, \quad \bar{T} = \frac{T}{\mu}, \quad \bar{u} = \frac{u}{L}, \quad \bar{\nu} = \frac{\nu}{L}\sqrt{\frac{\mu}{\rho}}, \tag{2.3}$$

where L is a characteristic length and μ is a constant with the dimension of stress. Differentiating both sides of (2.1) with respect to x, substituting (2.2) into the resulting equation and using (2.3), we obtain (1.1), where we drop the overbar for notational convenience. The question that we shall discuss throughout the rest of this work is which of the possible forms of the non-linear function $g(T)$ are relevant for the existence of traveling wave solutions of (1.1). Following mainly the standard techniques used widely in the literature to find traveling wave solutions we obtain the solutions of (1.1), explicitly, implicitly or numerically, for various forms of $g(T)$.

We now discuss some strain-limiting models reported in the literature for elastic and viscoelastic solids. The following is a list of non-linear constitutive relations $g(T)$ which we adopt in this study:

Model A: We first consider the one-dimensional version of the model proposed in an elastic setting by Kannan et al. in [9], namely,

$$g(T) = \beta T + \alpha\left(1 + \frac{\gamma}{2}T^2\right)^n T, \tag{2.4}$$

where $\alpha \geq 0$, $\beta \leq 0$, $\gamma \geq 0$ and n are constants. Note that when $n = 0$ and/or $\gamma = 0$, one recovers the standard constitutive equation for a linearized material. In Section 4, for the strain-limiting viscoelastic model defined by (2.2)–(2.4) we obtain traveling wave solutions explicitly if $n = 1$ and implicitly if $n = -1/2$.

Model B: The second model is based on a simplified version of the non-linear constitutive relation proposed by Rajagopal in [14]:

$$g(T) = \frac{T}{(1 + |T|^r)^{1/r}}, \tag{2.5}$$

where $r > 0$ is a constant. This model was studied in elastic settings by many authors in different contexts (see e.g. [1–3]). Note that when $\beta = 0$, $n = -1/2$, $\alpha = 1$ and $\gamma = 2$, Model A becomes equivalent to Model B with $r = 2$. In Section 4, when $r = 2$, traveling wave solutions corresponding to this model are obtained in closed form.

Model C: This model is the one-dimensional form of the constitutive relation proposed by Rajagopal in [13,14]:

$$g(T) = \alpha\left\{\left[1 - \exp\left(-\frac{\beta T}{1 + \delta|T|}\right)\right] + \frac{\gamma T}{1 + |T|}\right\}, \tag{2.6}$$

where α, β, γ and δ are constants. Note that when $\beta = 0$ and $\alpha = \gamma = 1$ this model reduces to Model B with $r = 1$. In Section 4, we solve the non-linear differential equation corresponding to this model numerically and compute traveling wave solutions for a specific set of parameter values.

Model D: This model is the one-dimensional form of a different model again introduced by Rajagopal in [13,14]:

$$g(T) = \alpha\left(1 - \frac{1}{1 + \dfrac{T}{1 + \delta|T|}}\right) + \beta\left(1 + \frac{1}{1 + \gamma T^2}\right)^n T, \tag{2.7}$$

where α, β, γ and δ are constants. Note that when $\alpha = 0$, with appropriate choice of the remaining parameters, we may derive Model A from this model. In Section 4, traveling wave solutions corresponding to this model are also obtained numerically for a specific set of parameter values.