International Journal of Non-Linear Mechanics 77 (2015) 74–84

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Extended thermodynamics for dense gases up to whatever order

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ARTICLE INFO

Article history:
Received 25 March 2015
Received in revised form
7 July 2015
Accepted 28 July 2015
Available online 6 August 2015

Keywords:
Extended thermodynamics
Dense gas
Moments equations

ABSTRACT

The 14 moments model for dense gases, introduced in the last years by Arima, Taniguchi Ruggeri, Sugiyama, is here considered. They have found the closure of the balance equations up to second order with respect to equilibrium; here the closure is found up to whatever order with respect to equilibrium, but for a more constrained system where more symmetry conditions are imposed and this in agreement with the suggestion of the kinetic theory. The results, when restricted at second order with respect to equilibrium, are the same of the previously cited model but under the further restriction of full symmetries.

1. Introduction

Starting point of this research is the paper [1] which belongs to the framework of Extended Thermodynamics. Some of the original papers on this subject are [2,3] while more recent papers are [4–15] and the theory has the advantage to furnish hyperbolic field equations, with finite speeds of propagation of shock waves and very interesting analytical properties.

It starts from a given set of balance equations where some arbitrary functions appear; restrictions on these arbitrariness are obtained by imposing the Entropy Principle and the relativity principle.

However, these restrictions were very strong; for example, the internal energy ϵ was not independent of the pressure p but linked to it by the relation $2\rho\epsilon = 3p$, where ρ is the mass density.

This drawback has been overcome in [1] and other articles such as [16–32] by considering two blocks of balance equations; for example, in the 14 moments case treated in [1], they are

$$\partial_t F^M + \partial_k F^{kM} = P^M, \quad \partial_t G^E + \partial_k G^{kE} = Q^E, \tag{1}$$

where

$$F^M = (F, F^i, F^{ij}), \quad G^E = (G, G^i),$$
$$F^{kM} = (F^k, F^{ki}, F^{kij}), \quad G^{kE} = (G^k, G^{ki}),$$
$$P^M = (0, 0, P^{ij}), \quad Q^E = (0, Q^i).$$

The first 2 components of P^M are zero because the first 2 components of equations $(1)_1$ are the conservation laws of mass and momentum; the first component of Q^E is zero because the first component of equations $(1)_2$ is the conservation laws of energy. The whole block $(1)_2$ can be considered an "Energy Block" and we have used the indexes M and E to distinguish quantities belonging to the "Mass Block" from quantities belonging to the "Energy Block".

Eqs. (1) can be written in a more compact form as

$$\partial_t F^A + \partial_k F^{kA} = P^A, \tag{2}$$

where

$$F^A = (F^M, G^E), \quad F^{kA} = (F^{kM}, G^{kE}), \quad P^A = (P^M, Q^E).$$

In the whole set (2), F^A are the independent variables, while F^{kij}, G^{ki}, P^{ij}, Q^i are constitutive functions. Restrictions on their generalities are obtained by imposing

1. *The Entropy Principle* which guarantees the existence of an entropy density h and an entropy flux h^k such that the equation

   $$\partial_t h + \partial_k h^k = \sigma \geq 0, \tag{3}$$

   holds whatever solution of the Eqs. (2).
   Thanks to Liu' s Theorem [33,34], this is equivalent to assuming the existence of Lagrange Multipliers μ_A such that

   $$dh = \mu_A dF^A, \quad dh^k = \mu_A dF^{kA}, \quad \sigma = \mu_A P^A. \tag{4}$$

   An idea conceived by Ruggeri in [35] is to define the 4-potentials h', h'^k as

   $$h' = \mu_A F^A - h, \quad h'^k = \mu_A F^{kA} - h^k, \tag{5}$$

   so that Eqs. $(4)_{1,2}$ become

   $$dh' = F^A d\mu_A, \quad dh'^k = F^{kA} d\mu_A,$$

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http://dx.doi.org/10.1016/j.ijnonlinmec.2015.07.011
0020-7462/

which are equivalent to

$$F^A = \frac{\partial h'}{\partial \mu_A}, \quad F^{kA} = \frac{\partial h'^k}{\partial \mu_A}, \tag{6}$$

if the Lagrange Multipliers are taken as independent variables. A nice consequence of Eqs. (6) is that the field equations assume the symmetric form.

Other restrictions are given by

2. *The symmetry conditions*, that is the second component of F^M is equal to the first component of F^{kM}, the third component of F^M is equal to the second component of F^{kM}, the second component of G^E is equal to the first component of G^{kE}. Moreover, F^{ij}, F^{kij} and G^{ki} are symmetric tensors. The symmetry of F^{kij} and G^{ki} is motivated by the kinetic counterpart of this theory (see Ref. [23]), even if was not imposed in [1] in order to have a more general model. We propose, in a future article, to remove this further constraint. Thanks to Eqs. (6) these conditions may be expressed as

$$\frac{\partial h'}{\partial \mu_i} = \frac{\partial h'^i}{\partial \mu}, \quad \frac{\partial h'}{\partial \mu_{ij}} = \frac{\partial h'^i}{\partial \mu_j}, \quad \frac{\partial h'}{\partial \lambda_i} = \frac{\partial h'^i}{\partial \lambda}, \quad \frac{\partial h'^{[k}}{\partial \mu_{i]j}} = 0, \quad \frac{\partial h'^{[k}}{\partial \lambda_{i]}} = 0, \tag{7}$$

where we have assumed the decomposition $\mu_A = (\mu, \mu_i, \mu_{ij}, \lambda, \lambda_i)$ for the Lagrange Multipliers. Moreover μ_{ij} is a symmetric tensor. The next conditions come from

3. *The Galilean relativity principle*: There are two ways to impose this principle. One of these is to decompose the variables F^A, F^{kA}, P^A, μ_A in their corresponding non-convective parts $\hat{F}^A$, $\hat{F}^{kA}$, $\hat{P}^A$, $\hat{\mu}_A$ and in velocity dependent parts, where the velocity is defined by $v^i = F^{-1}F^i$. After that, all the conditions are expressed in terms of the non-convective parts of the variables. This procedure is described in [2,34] for the case considering only the block $(1)_1$ and is followed in [1] for the whole set (1).

Another way to impose this principle leads to easier calculations; it is described in [36] for the case considering only the block $(1)_1$ and in [32] for the 18 moments model formulated in the framework of the theory with both blocks. The resulting Eqs. (13) and (14) of [32] contain two additional variables with respect to the present model, that is μ_{ill} and λ_{ll}; by putting these variables equal to zero, we obtain the counterpart for our model, that is

$$\begin{aligned}
&\frac{\partial h'}{\partial \mu}\mu_i + \frac{\partial h'}{\partial \mu_h}(2\mu_{ih} + 2\lambda\delta_{hi}) + 2\frac{\partial h'}{\partial \mu_{hi}}\lambda_h + \frac{\partial h'}{\partial \lambda}\lambda_i = 0,\\
&\frac{\partial h'^k}{\partial \mu}\mu_i + \frac{\partial h'^k}{\partial \mu_h}(2\mu_{ih} + 2\lambda\delta_{hi}) + 2\frac{\partial h'^k}{\partial \mu_{hi}}\lambda_h + \frac{\partial h'^k}{\partial \lambda}\lambda_i + h'\delta^{ki} = 0.
\end{aligned} \tag{8}$$

To be more precise, the Galilean invariance conditions, as shown in Ref. [37], dictate a precise velocity dependence of all the moments, of the entropy density and of the entropy flux density; from them it follows the dependence on velocity of the Lagrange multipliers and of the 4-potential. In particular, the independence of h' and $h'^k - h'v^k$ on velocity is expressed by Eqs. (8). To obtain them we have used the dependence on velocity reported in [1]. Now, the methodology exposed in [36] consists in imposing firstly only the equations (8) for the Galilean Relativity Principle supported by the Entropy Principle; the dependence on velocity is imposed in a second step, when we come back from the Lagrange multipliers as variables to the non-convective parts of the moments. At this stage all the difficult calculations which we have apparently avoided, now present themselves again as it can be seen also in the present article for the passages of Sections 3 and 4. But they impose no further conditions because they amount only in an inversion of variables and in the solving problem of implicit functions; so these calculations are very difficult, but straightforward.

So we have to find the most general functions satisfying (7) and (8). After that, we have to use Eqs. $(6)_1$ to obtain the Lagrange Multipliers in terms of the variables F^A. By substituting them in $(6)_2$ and in $h' = h'(\mu_A)$, $h'^k = h'^k(\mu_A)$ we obtain the constitutive functions in terms of the variables F^A. If we want the non-convective parts of our expressions, it suffices to calculate the left hand side of Eqs. $(6)_1$ in $\vec{v} = \vec{0}$ so that they become

$$\hat{F}^A = \frac{\partial h'}{\partial \mu_A}. \tag{9}$$

From this equation we obtain the Lagrange Multipliers in terms of $\hat{F}^A$ (Obviously, they will be $\hat{\mu}_A$) and after that substitute them in $h' = h'(\mu_A)$, $h'^k = h'^k(\mu_A)$ (the last of which will in effect be $\hat{h}'^k$) and into $\hat{F}^{kA} = \frac{\partial h'^k}{\partial \mu_A}$, that is Eq. $(6)_2$ calculated in $\vec{v} = \vec{0}$.

It has to be noted that from $v^i = F^{-1}F^i$ it follows $\hat{F}^i = 0$, so that one of the equations (9) is $0 = \frac{\partial h'}{\partial \mu_i}$; this does not mean that h' does not depend on μ_i, but this is simply an implicit function defining jointly with the other equations (9) the quantities $\hat{\mu}_A$ in terms of $\hat{F}^A$. We note also here the ground to settle $\mu_i = 0$ at equilibrium: in fact, in this state we have $\mu_{ij} = 0$, $\lambda_i = 0$ so that, for the Representation Theorems, $\frac{\partial h'}{\partial \mu_i}$ is proportional to μ_i and $\frac{\partial h'}{\partial \mu_i} = 0$ implies $\mu_i = 0$.

By using a procedure similar to that of the paper [36], we can prove that we obtain the same results of the firstly described approach.

Now, from $(7)_2$ it follows $\frac{\partial h'^{[i}}{\partial \mu_{j]}} = 0$; this equation, together with $(7)_1$ are equivalent to assuming the existence of a scalar function H such that the following equation holds:

$$h' = \frac{\partial H}{\partial \mu}, \quad h'^i = \frac{\partial H}{\partial \mu_i}. \tag{10}$$

In fact, the integrability conditions for (10) are exactly $(7)_1$ and $\frac{\partial h'^{[i}}{\partial \mu_{j]}} = 0$.

Thanks to (10), we can rewrite (7) and (8) as

$$\frac{\partial^2 H}{\partial \mu \partial \mu_{ij}} = \frac{\partial^2 H}{\partial \mu_i \partial \mu_j}, \quad \frac{\partial^2 H}{\partial \mu \partial \lambda_i} = \frac{\partial^2 H}{\partial \lambda \partial \mu_i}, \quad \frac{\partial^2 H}{\partial \mu_{[k} \partial \mu_{i]j}} = 0, \quad \frac{\partial^2 H}{\partial \mu_{[k} \partial \lambda_{i]}} = 0. \tag{11}$$

$$\begin{aligned}
&\frac{\partial^2 H}{\partial \mu^2}\mu_i + \frac{\partial^2 H}{\partial \mu \partial \mu_h}(2\mu_{ih} + 2\lambda\delta_{hi}) + 2\frac{\partial^2 H}{\partial \mu \partial \mu_{hi}}\lambda_h + \frac{\partial^2 H}{\partial \mu \partial \lambda}\lambda_i = 0,\\
&\frac{\partial^2 H}{\partial \mu \partial \mu_k}\mu_i + \frac{\partial^2 H}{\partial \mu_h \partial \mu_k}(2\mu_{ih} + 2\lambda\delta_{hi}) + 2\frac{\partial^2 H}{\partial \mu_k \partial \mu_{hi}}\lambda_h + \frac{\partial^2 H}{\partial \mu_k \partial \lambda}\lambda_i + \frac{\partial H}{\partial \mu}\delta^{ki} = 0.
\end{aligned} \tag{12}$$

We note now that the derivative of $(12)_1$ with respect to μ_k is equal to the derivative of $(12)_2$ with respect to μ; similarly, the derivative of $(12)_1$ with respect to λ_k is equal to the derivative of $(12)_2$ with respect to λ, as it can be seen by using also Eqs. (11).

Consequently, the left hand side of Eq. $(12)_1$ is a vectorial function depending only on two scalars μ, λ and on a symmetric tensor μ_{ij}. For the Representation Theorems [38–41], it can be only zero.

In other words, Eq. $(12)_1$ is a consequence of (11) and $(12)_2$, so that it has not to be imposed. By using Eqs. (11) we can rewrite Eq. $(12)_2$ as

$$\frac{\partial^2 H}{\partial \mu \partial \mu_k}\mu_i + 2\frac{\partial^2 H}{\partial \mu \partial \mu_{kj}}\mu_{ji} + 2\frac{\partial^2 H}{\partial \mu \partial \mu_{ki}}\lambda + 2\frac{\partial^2 H}{\partial \mu_k \partial \mu_{ij}}\lambda_j + \frac{\partial^2 H}{\partial \mu \partial \lambda_k}\lambda_i + \frac{\partial H}{\partial \mu}\delta^{ki} = 0. \tag{13}$$

In Section 2 of the present paper, we find the general solution up to whatever order with respect to equilibrium, of the conditions (11) and (13). This is not a trivial aspect; in fact, if we impose the conditions up to a given order n with respect to equilibrium, there is always the risk to obtain restrictions on the same expressions by imposing the equations at an higher order! With the present study this risk is eliminated.

In Section 3 we will see the implications of this solution to a second order theory, coming back to the moments as independent