International Journal of Non-Linear Mechanics 77 (2015) 107–114

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Limit point instability in pressurization of anisotropic finitely extensible hyperelastic thin-walled tube

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ARTICLE INFO

Article history:
Received 3 July 2014
Received in revised form
3 August 2015
Accepted 3 August 2015
Available online 12 August 2015

Keywords:
Anisotropy
Hyperelasticity
Limiting fiber extensibility
Stability
Thin-walled tube
Pressure

ABSTRACT

Mechanical responses of materials undergoing large elastic deformations can exhibit a loss of stability in several ways. Such a situation can occur when a thin-walled cylinder is inflated by an internal pressure. The loss of stability is manifested by a non-monotonic relationship between the inflating pressure and internal volume of the tube. This is often called limit point instability. The results, known from the literature, show that isotropic hyperelastic materials with limiting chain extensibility property always exhibit a stable response if the extensibility parameter of the Gent model satisfies $J_m < 18.2$. Our study investigates the same phenomenon but for tubes with anisotropic form of the Gent model (finite extensibility of fibers). Anisotropy, used in our study, increases the number of material parameters the consequence of which is to increase degree of freedom of the problem. It will be shown that, in stark contrast to isotropic material, the unstable response is predicted not only for large values of J_m but also for $J_m \approx 1$ and smaller, and that the existence of limit point instability significantly depends on the orientation of preferred directions and on the ratio of linear parameters in the strain energy density function (this ratio can be interpreted as the ratio of weights by which fibers and matrix contribute to the strain energy density). Especially tubes reinforced with fibers oriented closely to the longitudinal direction are susceptible to a loss of monotony during pressurization.

1. Introduction

Materials undergoing large elastic deformations exhibit many non-self-evident phenomena in their mechanical behavior. Some of them are linked to a loss of stability during deformation. A non-monotonic inflation of cylinders and spheres, kink formation on a twisted rod, or wrinkling of the surface of the bent block may be mentioned as examples [15,16,18,40,12]. In the past, the onset of instability and related deformation modes have been frequently discussed with reference to rubber-like materials which, due to their non-linear mechanical behavior and wide industrial use, significantly stimulated development of the modern theory of elasticity. Increasing knowledge of the mechanics of soft tissues (arteries, veins, muscles, skin, tendons, ligaments, and esophagus), which inherently undergo large deformations and exhibit material non-linearity, has also significantly contributed to the progress in non-linear elasticity.

In biomechanics, a loss of deformation stability has been hypothesized to be a possible explanation for non-physiological sates such as arterial aneurysm formation [1,55,16,54,40,20], blood vessel tortuosity and buckling [24,5,13,22,23,43,2,3], and formation of skin wrinkles [11,12]. Despite the negatives, loss of deformation stability can also play a positive role and can be physiologically advantageous. For instance, the collapse of lower limb veins during skeletal muscle contraction, which helps move blood towards the heart against gravity [21,56,57], is a perfect example. This mechanism is better known as the skeletal-muscle pump.

Elastic instability can be understood from two different viewpoints [18]. The first approach, dealing with limit point instability, focuses on the existence of a local extreme in the mechanical response of the material [15,16,18,40]. This situation is well-known from an inflation of balloons and cylindrical tubes made from an elastomer. Their mechanical response is characterized by high initial resistance to inflation. However, as pressurization continues, pressure increments necessary for radial expansion decrease and subsequently a maximum pressure is reached. Material response follows with decreasing pressure although the deformation still increases. Maximum pressure may be either local or global maximum depending on specific constitutive model of the material.

From the second viewpoint, the instability is understood as a bifurcation of the solution of a boundary value problem. In the

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http://dx.doi.org/10.1016/j.ijnonlinmec.2015.08.003
0020-7462/

example of the inflation of a cylindrical tube, the bifurcation of the solution means that deformed tube may experience configurations different from ordinary uniform radial expansion. The bulging (the tube locally expands in the radial direction), the loss of circularity of the cross-section, and the buckling (the tube deflects in a way that is similar to a long slender column under compression) may appear during pressurization [10,41,26,55,2,3,44,52]. More complex deformation modes such as a local bulge on a deflected tube [17,42] may also appear. Deformed configurations corresponding to bifurcated solutions are obtained by solving the problem formulated by means of the theory of incremental deformations [6], where small (localized) displacements are superimposed on a finite solution [27,18,26,48,55]. In other words, bifurcated solutions are located in close neighborhood of the finite solution [18].

In the present paper, we will focus on the "limit point instability" phenomenon occurring in the inflation of hyperelastic thin-walled cylindrical tubes. The motivation for our study came from an article entitled *Elastic instabilities for strain-stiffening rubber-like spherical and cylindrical thin shells under inflation*, which was published by Kanner and Horgan [40].

Kanner and Horgan [40] investigated the existence of limit point instability in thin-walled hyperelastic tubes with material properties described by Gent's elastic potential [14,34–37]. The Gent hyperelastic model is phenomenological analog to the *limiting* (*finite*) *chain extensibility* models derived from the concept of finitely extensible macromolecular chains within the framework of the statistical theory of elastomers [4,7,8,50]. These models are especially suitable to describe mechanical response characterized by rapid strain stiffening, which is for instance, the case of soft tissues.

Kanner and Horgan [40] showed that, depending on the specific value of the limiting extensibility parameter J_m of the Gent model, non-monotonic inflation of thin-walled cylindrical tubes may or may not occur. To be more specific, the inflation is always stable for $J_m < 18.2$. In other words, tubes made from rapidly stiffening material do not exhibit limit point instability. Kanner and Horgan [40] concluded that the known numerical values of J_m for arterial tissue correspond to stable mechanical behavior which suggests that inflation instability is not a mechanism contributing to aneurysm pathophysiology.

The form of the Gent model used by Kanner and Horgan [40] is isotropic. Nevertheless, it is now widely accepted that anisotropic constitutive models should be used in studies describing blood vessels [29,31]. In the present study, limit point instability during inflation of the cylindrical tube will be analyzed, but in contrast to Kanner and Horgan [40], the material of the tube will not be isotropic. It will be anisotropic analog to Gent's elastic potential. Particular form of the strain energy density function used in our study is based on *limiting fiber extensibility*, a concept introduced by Horgan and Saccomandi [36] and utilized by Horný et al. [39] in the estimation of material parameters for the human abdominal aorta. It will be shown that, in stark contrast to isotropic material, the unstable response is predicted not only for large values of J_m but also for $J_m \approx 1$ and smaller, and that the existence of limit point instability significantly depends on the orientation of preferred directions and on the ratio of linear parameters in the strain energy density function, which can be interpreted as the ratio of fiber-to-matrix weights.

2. Constitutive model

The material of the tube will be considered to be incompressible and hyperelastic characterized by the strain energy function W defined per unit reference volume. In such a case the constitutive equation can be written in the form (1), Holzapfel [28]. Here $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor. $\mathbf{F}$ is the deformation gradient defined as $\mathbf{F}=\partial\boldsymbol{x}/\partial\boldsymbol{X}$, where $\boldsymbol{x}$ and $\boldsymbol{X}$, respectively, denote the position vector of a material particle in the deformed and the reference configuration. p plays the role of a Lagrangian multiplier which represents the hydrostatic contribution to $\boldsymbol{\sigma}$, not captured by W, due to the incompressibility constraint

$$\boldsymbol{\sigma} = -p\mathbf{I} + \frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^T \tag{1}$$

Over the last decades, several models for W have been proposed to describe the mechanical behavior of elastomers and soft tissues under large strains. The classic approach, proposed by R.S. Rivlin, was to suggest that W is a polynomial function of strain invariants. Such functions, though suitable for elastomers, do not appropriately describe mechanical behavior of soft tissues exhibiting rapid strain stiffening [29,30]. It has been recently shown that models based on the concept of *limiting chain extensibility* are suitable for both elastomers and soft tissues [35,51,38,39]. In their study focused on inflation stability, Kanner and Horgan [40] investigated a model of W based on this concept. More specifically, it was a phenomenological analog proposed by Gent [14], W_G. Its mathematical form is expressed in the following equation:

$$W_G = -\frac{\mu J_m}{2}\ln\left(1-\frac{I_1-3}{J_m}\right) \tag{2}$$

Here μ is the shear modulus at infinitesimal strains. I_1 denotes the first invariant of the right Cauchy-Green strain tensor $\mathbf{C}$ ($I_1 = trace(\mathbf{C})$), where $\mathbf{C}=\mathbf{F}^T\mathbf{F}$. J_m is referred to as the *limiting extensibility* (dimensionless) *parameter* because it restricts admissible deformations of the material to the domain where $I_1 - 3 < J_m$ applies. In other words, $I_1 - 3 \to J_m^-$ implies $W \to \infty$. Thus finite extensibility of a macromolecular chain is, using the phenomenological approach, captured by a suitable mathematical form of the strain energy (logarithmic function). It is also clear that this mathematical choice gives the stress–strain relationship that in some intervals increases more rapidly than the polynomial and exponential functions. Regarding inflation instability, Kanner and Horgan [40] found that there is a critical value of J_m, $J_m = 18.2$, which discriminates the behavior of a pressurized thin-walled cylindrical tube. The inflation is stable (monotonically increasing pressure for increasing circumferential stretch) for materials with $J_m < 18.2$. Whereas for materials with $J_m > 18.2$, there is a local maximum followed by local minimum that is subsequently followed by a steeply increasing section of the pressure–stretch curve (see [40, Figure 5]).

In contrast to industrial rubber-like elastomers, there is only a limited extent to which isotropic material models can be utilized for biological tissues. It is well known that although they are inherently macromolecular similarly to elastomers, soft tissues exhibit more or less anisotropic behavior, which results from their complicated hierarchical structure where amino acids are aggregated into polypeptide chains, the chains into specific protein macromolecules, the proteins into fibrils, and finally the fibrils are arranged into fibers that are visible on a microscopic scale (i.e. there are four orders of spatial arrangement; [9]). Horgan and Saccomandi [36] recently proposed an anisotropic constitutive model that was inspired by the *limiting chain extensibility* concept. The specific form of strain energy density is shown in the following equation:

$$W_{HS} = \frac{\mu}{2}(I_1-3) - \sum_{k=4,6}\frac{\nu J_m}{2}\ln\left(1-\frac{(I_k-1)^2}{J_m}\right) \tag{3}$$

The model (3) is composed of the neo-Hookean part, which can be interpreted as the contribution of the isotropic matrix, and two logarithmic terms that depend on the deformation invariants I_4 and I_6. I_4 and I_6 are defined as the square of the stretch of the unit referential vectors $\boldsymbol{M}$ and $\boldsymbol{N}$ aligned with the preferred directions