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Discrete and non-local *elastica*

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ABSTRACT

In this paper, the buckling and post-buckling behavior of an elastic lattice system referred to as the discrete *elastica* problem is investigated using an equivalent non-local continuum approach. The geometrically exact post-buckling analysis of the elastic chain, also called Hencky system, is first numerically solved using the shooting method. This discrete physical model is also mathematically equivalent to a finite difference formulation of the continuum *elastica*. Starting from the exact difference equations of the discrete problem, a continualization method is applied for approximating the difference operators by differential ones, in order to better characterize the discrete system by an enriched continuous one. It is shown that the new continuum associated with the discrete system exactly fits the discrete *elastica* post-buckling problem, where the non-locality is of Eringen's type (also called stress gradient non-local model). An asymptotic expansion is performed for both the discrete and the non-local continuum models, in order to approximate the post-buckling branches of the discrete system. Some numerical investigations show the efficiency of the non-local approach, especially for capturing the scale effects inherent to the cell size of the lattice model.

1. Introduction

The transition from discrete to continuous nature of structural morphology is of primary interest in physics for understanding how the microstructure may influence the macroscopic properties of the material on a larger scale. Matter is intimately discrete at a finer scale, especially at the atomistic scale, while engineering models are usually predicated on some continuous assumptions for efficient computational cost and mathematical investigations. There is already a large body of literature devoted to the transition between lattice systems to continuous systems, especially for the linear range (see for instance [1]). However, most of the reported results in this area are confined to the linear range and are restricted to some so-called local constitutive laws for the equivalent continuum, without any possibilities for capturing properly the transition from the discrete to the continuum nature of the material. This paper aims to contribute to a better understanding of the transition from a discrete to a continuous modeling in the presence of geometrical non-linearities. Both discrete and continuous models are formulated into a geometrically exact framework. The buckling and post-buckling behavior of an elastic lattice system referred to as herein as the discrete *elastica* problem is investigated via an equivalent non-local continuum approach.

The investigation of buckling and post-buckling behaviors of inextensible elastic columns (linearized problem and geometrically exact formulation) dates back to the mid 1700s [2]. On the other hand, the discrete elastic bar-chain problem was only analysed in the early part of the 1900s. A discrete formulation of Euler's problem was first studied by Hencky in 1920 [3] who considered an elastic bar-chain composed of rigid links connected by rotational springs. Hencky [3] showed that this system may asymptotically converge towards the Euler one for an infinite number of links. Note that Hencky did not provide the analytical solutions of the buckling load for an arbitrary number of links n, but he presented solutions only for $n=2$, 3 or 4. This problem has been reconsidered by Wang [4,5] who gave the buckling solution for any number of links n. In fact, Wang [4,5] solved a linear second-order difference equation (see for instance [6] for a general overview of difference equations) and analytically obtained the buckling load associated with the corresponding boundary value problem. Silverman [7] also mentioned the mathematical analogy between Hencky's system and the central finite difference formulation of Euler's problem. This buckling problem was reconsidered by Seide [8] in terms of the finite difference method (which may be regarded as equivalent to the algebraic equations of Hencky system), and compared to other numerical methods for general boundary

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conditions. Such a paradigmatic structural problem was reconsidered by Gáspár and Domokos [9], Domokos [10] or Domokos and Holmes [11] who pointed out the very rich structure inherent to the discrete property of the structural system, and the possible spatial chaotic behavior of Hencky bar-chain to be far away from the first initial bifurcated branch. Domokos and Holmes [11] also showed the mathematical equivalence of the finite difference formulation of Euler problem and the Hencky system in the non-linear range. It is worth mentioning that the spatial chaotic behavior of discrete chains has been also observed for non-conservative systems by Kocsis and Károlyi [12]. More recently, Wang et al. [13] or Challamel et al. [14,15] showed the possibility of linking the linearized Hencky system to non-local mechanics. Non-local mechanics is understood herein as an Eringen's type non-local differential model [16] where an implicit differential equation is assumed between the stress and the strain (or the bending moment and the curvature at the beam scale). This non-local model is also known as a stress gradient non-local model.

The non-local methodology has been mainly applied to the linearized problem, and one open question remains to be addressed, i.e. whether it can be applied in the non-linear range as well? To answer this question, we consider the non-local *elastica* and find out if it is able to capture the scale effects of Hencky chain, i.e. the discrete *elastica*. It is worth mentioning that the non-local *elastica* has been already investigated analytically or numerically by Wang et al. [17] or Shen [18] for perfectly straight non-local columns, or Xu et al. [19] who considered some initial imperfections. Shen [18] also performed some asymptotic expansions in order to obtain some approximated analytical expressions for the post-bifurcated branch. The optimization of non-local elastic columns has been also recently reported by Atanackovic et al. [20].

The paper is organized as follows. The geometrically exact post-buckling analysis of the elastic chain, also called Hencky system, is first numerically solved by using the shooting method. This discrete physical model is also mathematically equivalent to a finite difference formulation of the continuum *elastica*. Starting from the exact difference equations of the discrete problem, a continualization method is then applied for approximating the difference operators by differential ones, in order to better characterize the discrete system by an enriched continuous one. It is shown that the new continuum associated with the discrete system exactly fits the initial lattice problem, where the non-locality is of Eringen's type, also called stress gradient non-local model. An asymptotic expansion applied to the non-local *elastica* is performed for approximating the post-buckling branches of the discrete model. The analytical results of Shen [18] are found again for the non-local *elastica* and fit well the numerical results issued of the discrete system with the correct length scale calibration. We also presented an asymptotic expansion directly applied to the discrete *elastica*, which is not available in the literature, to the authors' knowledge. Some numerical investigations show the efficiency of the non-local approach, especially for capturing the scale effects inherent to the cell size of the lattice model. It is worth noting that there is an analogy between the elastic pendulum equation (dynamics initial value problem) and the *elastica* equation (static boundary value problem). This Kirchhoff analogy (see [21]; see also the discussion in [22,23] or [11]) triggers us to believe that the non-local spatial concepts developed herein can be probably generalized to a kind of time non-locality for the dynamics problem.

2. Discrete *elastica*

Consider a Hencky's bar-chain with pinned–pinned ends as shown in Fig. 1. The column, composed of n repetitive cells of size denoted by a, is axially loaded by a concentrated force denoted by

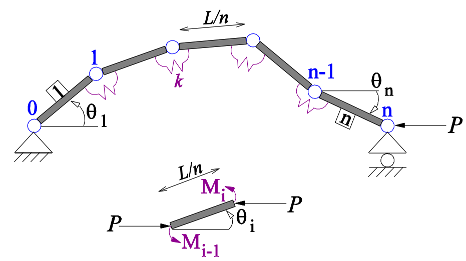

Fig. 1. Hencky's chain: n rigid links are connected by hinges and rotational springs.

P. The discrete column of length L is modeled by some finite rigid segments and elastic rotational springs of stiffness $k = EI/a$, where EI is the bending rigidity of the local Euler–Bernoulli column asymptotically obtained for an infinite number n of rigid links. In other words, the total length of the structure L is equal to $L = n \times a$, the number of rigid segments multiplied by the size of each segment.

The discrete version of the *local elastica* can be obtained from the following system of non-linear difference equations:

$$M_i = EI\frac{\theta_{i+1}-\theta_i}{a} \quad \text{and} \quad \frac{M_i - M_{i-1}}{a} + P\sin\theta_i = 0 \tag{1}$$

Here M_i is the bending moment in the rotational spring at hinge i, and θ_i is the angle of the ith link from the line of action of compressive force P. In other words, θ_i is the rotation angle of the segment i connecting the $(i-1)$th and the ith nodes.

As pointed out by Domokos and Holmes [11], these difference equations (1) are similar to the forward and the backward finite difference equations of the continuous elastic problem, where the step size is equal to the length of the rigid link a. In this concept, the differential equation system of the axially compressed, hinged-hinged *elastica*, $M = EI \times d\theta/ds$ and $dM/ds + P\sin\theta = 0$, are discretized using forward and backward differences, respectively. This yields a semi-implicit Euler method, which defines an area preserving map.

The non-linear second-order difference equation is obtained from Eq. (1):

$$EI\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{a^2} + P\sin\theta_i = 0 \tag{2}$$

This non-linear difference equation is reformulated in a dimensionless form

$$\theta_{i+1}-2\theta_i+\theta_{i-1} = -\frac{\beta}{n^2}\sin\theta_i \tag{3}$$

by using the dimensionless load $\beta = PL^2/EI$. The non-linear difference equation can be equivalently reformulated with the following relations:

$$\theta_{i+1} = \theta_i + \frac{\hat{\kappa}_i}{n} \quad \text{and} \quad \hat{\kappa}_{i+1} = \hat{\kappa}_i - \frac{\beta}{n}\sin\theta_{i+1} \tag{4}$$

with the dimensionless curvature $\hat{\kappa}_i$ defined by $\hat{\kappa}_i = L\kappa_i$ and the curvature $\kappa_i = M_i/EI$. The boundary conditions of the hinged-hinged column are obtained from the vanishing of the bending moments at both ends, i.e. $M_0 = 0$ and $M_n = 0$:

$$\theta_1 = \theta_0 \quad \text{and} \quad \theta_{n+1} = \theta_n \tag{5}$$

Note that an equivalent system may be obtained by discretizing the differential equations of the local *elastica* with central differences:

$$M_i = EI\frac{\tilde{\theta}_{i+1/2}-\tilde{\theta}_{i-1/2}}{a} \quad \text{and} \quad \frac{M_{i+1/2}-M_{i-1/2}}{a} + P\sin\tilde{\theta}_i = 0 \tag{6}$$

leading to the same non-linear difference equation:

$$\tilde{\theta}_{i+1}-2\tilde{\theta}_i+\tilde{\theta}_{i-1} = -\frac{\beta}{n^2}\sin\tilde{\theta}_i \tag{7}$$

The equivalence between the two discretizations is obtained from the