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On the parametrically excited pendulum equation with a step function coefficient

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ABSTRACT

An elementary geometric method is established to study non-linear second order differential equations with step function coefficient

$x'' + a^2(t)g(x) = 0, \quad a(t) := a_k \quad \text{if } t_{k-1} \le t < t_k \quad (k \in \mathbf{N}),$

where $a_k > 0$, $t_k \nearrow \infty$ as $k \to \infty$. The equation is rewritten into a discrete dynamical system on the plane. The method is applied to the excited pendulum equation when $g(x) = \sin x$. Starting from the usual periodic model, the problem of parametric resonance (problem of swinging) is investigated. It will be pointed out that the realistic model of swinging is not a periodically excited system, instead swing is a self-oscillating system. Finally, the classical Oscillation Theorem is extended to the non-linear periodic pendulum equation

$$\psi'' + a^2(t) \sin \psi = 0,$$

$$a(t) = \begin{cases} \sqrt{\dfrac{g}{l-\varepsilon}} & \text{if } 2kT \le t < (2k+1)T, \\ \sqrt{\dfrac{g}{l+\varepsilon}} & \text{if } (2k+1)T \le t < (2k+2)T \quad (k \in \mathbf{Z}_+), \end{cases}$$

where g and l denote the constant of gravity and the length of the pendulum, respectively; $\varepsilon > 0$ is a parameter measuring the intensity of swinging

1. Introduction

Non-autonomous second order differential equations

$$x'' = f(t, x) \quad (x \in \mathbb{R}^n),$$

whose right-hand sides explicitly depend on time t, model mechanical systems either being perturbed by external forces changing in time or containing some parameters that can vary as functions of time independently of state variables. In the latter case the system is *parametrically excited*. A classical parametrically excited system describing the motion of the lunar perigee is Hill's equation [1–4]

$$x'' + a^2(t)x = 0, \tag{1}$$

where coefficient $a(t)$ is $2T$-periodic. Lyapunov [5] and Haupt [6,7] proved the famous Oscillation Theorem about the existence of $2T$- and $4T$-periodic solutions. Meissner [8] studied the case when the coefficient $a(t)$ is a piecewise constant function assuming two different values. This case is of special interest in technical applications [9] and control due to, among others, the bang–bang principles. As Hochstadt [10] pointed out, this case is also important because the conditions guaranteeing the existence of periodic solutions can be expressed by elementary functions. In [11] we have established an elementary geometric method that was suitable for proving not only its existence part but the complete Oscillation Theorem including the oscillation properties of the solutions [12, p. 214]. The method is constructive, it yields the solutions themselves.

The mathematical model of swinging is also a parametrically excited equation of the form

$$x'' + a^2(t) \sin x = 0 \tag{2}$$

with an appropriate step function $a(t)$. The swinger has to choose $a(t)$ so that the zero solution be unstable (parametric resonance). In this paper at first we establish the realistic model of optimal swinging.

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Then we show that our method can be used for generalizing the Oscillation Theorem [2] to the *non-linear* equation (2).

The paper is organized as follows. In Section 2 we extend the method in [11] to the non-linear equation of the form

$$x'' + a^2(t)g(x) = 0$$

with an arbitrary step function a. In Section 3 we show that the realistic model of swinging is not a periodically excited system, instead swing is a self-oscillating system. In Section 4 we give the extension of the Oscillation Theorem to the parametrically excited pendulum equation (2). Sections 3 and 4 used the same method established in Section 2, but they are independent of each other, they can be read in any order.

2. The method

2.1. The phase plane and the dynamics

Let a continuously differentiable function $g:\mathbb{R}\to\mathbb{R}$, two sequences $\{a_k\}_{k=1}^{\infty}$, $\{t_k\}_{k=1}^{\infty}$ be given such that $a_k>0$, $0<t_k<t_{k+1}$ for all $k\in\mathbb{N}$, and let $t_0{:=}0$. Consider the non-linear second order differential equation

$$x'' + a^2(t)g(x) = 0, \quad a(t){:=}a_k \text{ if } t_{k-1}\le t<t_k \quad (k\in\mathbb{N}). \tag{3}$$

It is easy to see that

$$\frac{1}{2}(x')^2 + a_k^2 G(x) = c_k \quad \left(G(x){:=}\int_0^x g(s)\,ds\right)$$

is a first integral of (3) on the interval $[t_{k-1},t_k)$. To make the left-hand side of the first integral independent of k, introduce the new variable $y=x'/a_k$. Eq. (3) is equivalent to the system

$$x' = a_k y, \quad y' = -a_k g(x) \quad (t_{k-1}\le t<t_k, \quad k\in\mathbb{N}) \tag{4}$$

having the first integral

$$H(x,y){:=}\frac{1}{2}y^2 + G(x) = c_k \quad (t_{k-1}\le t<t_k, \quad k\in\mathbb{N}), \tag{5}$$

therefore trajectories of the system on the plane (x,y) consist of pieces of curves $H(x,y)=c_k$ $(k\in\mathbb{N})$. However, while global trajectories $[0,\infty)\to\mathbb{R}^2$ of (3) on the plane (x,x') have to be continuous curves, those of (4) are not continuous in general. Now we describe the dynamics of (4) on the (x,y) plane.

Let a trajectory originate from the point (ξ_0,η_0); the solution of (4) starting from this point at t_0 is denoted by

$$(x(t),y(t)) = (x(t;t_0,\xi_0,\eta_0), y(t;t_0,\xi_0,\eta_0)).$$

The first piece of the trajectory is located on the curve $H(x,y)=\eta_0^2/2+G(\xi_0)$ and goes to the point

$$x(t_1-0){:=}\lim_{t\to t_1-0} x(t;t_0,\xi_0,\eta_0), \quad y(t_1-0)=\lim_{t\to t_1-0} y(t;t_0,\xi_0,\eta_0).$$

The second piece of the trajectory originates from a point (ξ_1,η_1) and is located on the curve $H(x,y)=\eta_1^2/2+G(\xi_1)$ with a properly chosen (ξ_1,η_1). Since the corresponding solution (x,x') of (3) have to be continuous on the interval $[t_0,t_2)$ at $t=t_1$, we have

$$\xi_1 = x(t_1) = x(t_1-0), \quad \eta_1 = y(t_1) = \frac{x'(t_1)}{a_2} = \frac{x'(t_1-0)}{a_2} = \frac{a_1}{a_2}y(t_1-0).$$

This means that the trajectory "jumps" at $t=t_1$ from the point $(x(t_1-0), y(t_1-0))$ to the point $(x(t_1-0), (a_1/a_2)y(t_1-0))$. Geometrically, this transformation is a contraction or a dilation of the measure a_1/a_2 on the plane (x,y) in the direction of axis y. After the jump the trajectory goes continuously to the point $x(t_2-0;t_1,\xi_1,\eta_1), y(t_2-0;t_1,\xi_1,\eta_1)$ along the curve $H(x,y)=\eta_1^2/2+G(\xi_1)$. Similar to the point t_1, at t_2 the trajectory jumps to the point $(\xi_2,\eta_2){:=}(x(t_2-0), (a_2/a_3)y(t_2-0))$. And so on, the trajectory repeats these two steps. The complete system describing the dynamics on $[0,\infty)$ reads as follows:

$$\begin{cases} x' = a_k y, \quad y' = -a_k g(x) \quad \text{if } t_{k-1}\le t<t_k, \\ x(t_k) = x(t_k-0), \quad y(t_k) = \dfrac{a_k}{a_{k+1}}y(t_k-0) \quad (k\in\mathbb{N}), \end{cases} \tag{6}$$

which is a differential equation with impulses. The motion starting from (ξ_0,η_0) at $t=0$ can be characterized by the sequence $(\xi_k,\eta_k)=(x(t_k),y(t_k))$ $(k\in\mathbb{Z}_+{:=}\{0,1,2,\ldots\})$ uniquely determined by $(\xi_0,\eta_0)=(x(0),y(0))$. For example, to know the long time behavior of motions it is enough to know the sequences $\{(\xi_k,\eta_k)\}_{k=0}^{\infty}$.

2.2. The jump in polar coordinates

We introduce the polar coordinates (r,φ) on the plane (x,y) by

$$x = r\cos\varphi, \quad y = r\sin\varphi \quad (r>0, -\infty<\varphi<\infty), \tag{7}$$

and express the linear transformation

$$C_\kappa : (x,y)\mapsto(x,\kappa y) \quad (0<\kappa<\infty) \tag{8}$$

in these coordinates.

Denote by $(r_C,\varphi_C)=(\rho(r,\varphi;\kappa), \phi(\varphi;\kappa))$ the image of the point (r,φ) in polar coordinates at the contraction-dilation (8). Then

$$\rho(r,\varphi;\kappa) = \sqrt{x^2+\kappa^2y^2} = r\sqrt{1+(\kappa^2-1)\sin^2\varphi} = f(\varphi;\kappa)r,$$

$$f(\varphi;\kappa){:=}\sqrt{1+(\kappa^2-1)\sin^2\varphi} \quad (\kappa>0; -\infty<\varphi<\infty).$$

For $\phi(\varphi;\kappa)$, we know that $\tan\phi(\varphi;\kappa)=\kappa y/x=\kappa\tan\varphi$ for $x\neq0$, consequently

$$\phi(\varphi;\kappa){:=}\begin{cases} \arctan(\kappa\tan\varphi) + \left[\dfrac{\varphi+\frac{\pi}{2}}{\pi}\right]\pi & \text{if } \varphi\neq(2k+1)\dfrac{\pi}{2}, \\ \varphi & \text{if } \varphi=(2k+1)\dfrac{\pi}{2} \quad (k\in\mathbb{Z}), \end{cases}$$

where $[x]$ denotes the integer part of $x\in\mathbb{R}$.

The following lemma summarizes the basic properties of functions f and ϕ.

Lemma 2.1. *1. For every $\kappa>0$ the function $f(\cdot;\kappa):[0,\infty)\to(0,\infty)$ is even and π-periodic; furthermore*

$$f\left(\phi(\varphi;\kappa);\frac{1}{\kappa}\right) = \frac{1}{f(\varphi;\kappa)} \quad (\varphi\in\mathbb{R}) \tag{9}$$

(see Fig. 1).

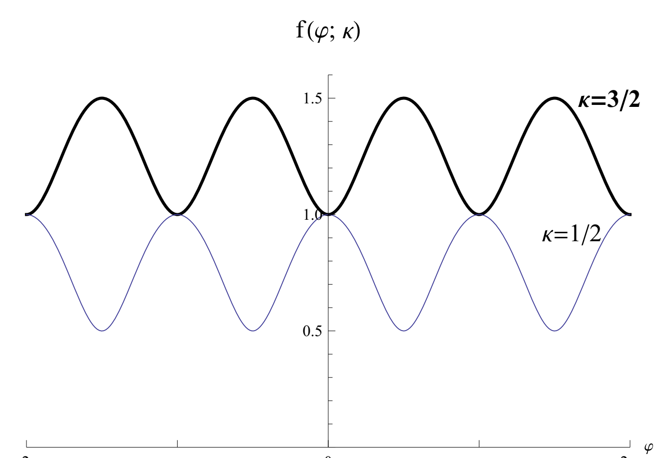

Fig. 1. Graphs of functions $f(\cdot;\kappa)$ for $\kappa<1$ and $\kappa>1$.