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The motion induced between radially extensional plates with one or both plates shrinking

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ABSTRACT

Flow between the radial extensional motion of parallel plates is studied for two cases. In the first case one plate stretches at rate a while the other plate shrinks at rate b and in the second case both plates shrink. Both cases can be considered by defining stretching ratio as $\sigma = b/|a|$. When both plates shrink one can find solutions in the region $\sigma < -1$ from those found in the region $-1 \le \sigma \le 0$. This feature is not available when one plate stretches and the other shrinks and thus σ must be varied over the region $\sigma \le 0$ to cover all possible solutions. Solutions are also dependent on a Reynolds number $R = |a|h^2/\nu$ where h is the plate separation distance and ν is the kinematic viscosity of the fluid. For both problems studied, we have determined the $R=0$ solutions and their large-R asymptotic behaviors. Using two numerical techniques, no bifurcated solutions were encountered. Results are presented in graphical form for the radial pressure gradient, lower and upper wall shear stresses, and velocity profiles for these axisymmetric flows. A region of zero wall shear stress exists for stretching/shrinking plates whilst the wall shear stresses for shrinking/shrinking plates are never zero. An interesting singular limit in solution behavior as $R \to \infty$ is found for the shrinking/shrinking plate flow.

1. Introduction

In a seminal paper, Crane [1] discovered an exact solution for the flow induced in the quiescent field above an impermeable flat plate linearly stretching along one coordinate axis. Many extensions of the problem have been reported including the effect of external flow above the plate and the effect of transpiration through a porous stretching plate; see the review article by Wang [2]. These stretching problems apply to many engineering processes such as hot rolling, wire drawing, glass-fiber production, etc; see Refs. [3–5].

Of particular interest is the paper by Wang [6] for the flow field above an impermeable orthogonally stretching flat plate, i.e. stretching with strain rate a along the x-coordinate and strain rate b along the y-coordinate. He solved the problem numerically for values of the strain rate ratio $\beta = b/a$ in the range $0 \le \beta \le 1$, where the value $\beta = 1$ represents a disk undergoing pure radial stretching. The radially stretching problem introduced in Wang [6] was extended by Fang [7] to include plate rotation. Recently, Weidman and Ishak [8] extended the work of Wang [6] to discover dual solutions for orthogonally stretching plates. The flow induced above sheared plates with and without stretching and stretching plates in a uniformly rotating system were studied by Weidman [9–11].

The above problems pertain to a single plate undergoing motion in its own plane beneath a quiescent fluid. These may be considered as *external flow* problems. The flow generated between parallel plates executing some form of motion may be considered as *internal flow* problems. Brady and Acrivos [12] were the first to consider the (equal) extensional flow of parallel plates in which several bifurcations and dual solutions were found. Yet a new transverse bifurcation of this problem was reported by Watson et al. [13]. In another effort, Fang and Zhang [14] and Van Gorder et al. [15] studied the flow contained between radially stretching disks. Unlike the two-dimensional case, only one solution was found. Such uniqueness may be supported by the studies of Nagayama and Okamoto [16]. The problem of flow between parallel plates with offset centers of equal strain rate stretching and shrinking was recently reported by Weidman [17].

It is to be noted that all problems reviewed above represent exact solutions of the Navier–Stokes equations according to the definition given by Drazin and Riley [18].

The goal of the present investigation is to extend the problem of radially stretching plates reported by Fang and Zhang [14] to the following two cases: (i) one disk stretches and the other shrinks, and (ii) both disks shrink. The presentation is as follows. In Section 2 the problem for both stretching/shrinking plates and shrinking/shrinking plates is presented. Results for the stretching/shrinking

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problem are given in Section 3 and those for the shrinking/shrinking problem are presented in Section 4. The paper concludes with a discussion of results and concluding remarks in Section 5.

2. Formulation

For axisymmetric flow without swirl using cylindrical coordinates (r_*, z_*) with velocities (u_*, w_*) we non-dimensionalize lengths with the plate separation distance h, meridional velocities with $|a|\,h$ where a is the radial stretch rate of the upper plate that may be either positive or negative, and pressure with $\mu|a|$ where μ is the absolute fluid viscosity. Then the motion is governed by the continuity equation and the meridional components of the Navier–Stokes equation:

$$R(uu_r + wu_\eta) = -p_r + (u_{rr} + \frac{1}{r}u_r + u_{\eta\eta} - \frac{u}{r^2}) \tag{2.1a}$$

$$R(uw_r + ww_\eta) = -p_\eta + (w_{rr} + \frac{1}{r}w_r + w_{\eta\eta}) \tag{2.1b}$$

in which

$$R = \frac{|a|\,h^2}{\nu}, \quad \eta = \frac{z_*}{h}. \tag{2.1c}$$

With the lower plate at $z_* = 0$ and the upper plate at $z_* = h$ the boundary conditions for impermeable plates are given as

$$u = \pm r, \quad w = 0 \quad (\eta = 1) \tag{2.2a}$$

$$u = \sigma r, \quad w = 0 \quad (\eta = 0), \quad \sigma = \frac{b}{|a|} \tag{2.2b}$$

where the plus and minus sign in (2.2a) denote radial stretching and shrinking, respectively, and b may be of either sign.

Inserting the similarity reduction

$$u = f'(\eta), \quad w = -2f(\eta) \tag{2.3}$$

into the above equations yields the ordinary differential equation:

$$f''' + R(2ff'' - f'^2) - \beta = 0. \tag{2.4}$$

The boundary conditions for impermeable lower and upper plates are

$$f(0) = 0, \quad f'(0) = \sigma, \quad f(1) = 0, \quad f'(1) = \pm 1. \tag{2.5}$$

The pressure field calculated from integration of the z-momentum equation gives

$$p(r, \eta) = \beta\frac{r^2}{2} - 2(Rf^2 + f'). \tag{2.6}$$

Evaluation of Eq. (2.4) at the upper and lower walls for each problem yields the relation

$$\beta = f'''(0) - R\sigma^2 = f'''(1) - R \tag{2.7}$$

which can be used as a check on numerical integrations.

We denote τ_r as the radial stress. Then the lower and upper wall shear stresses are given as

$$\tau_r(z_* = 0) = \mu\frac{\partial u_*}{\partial z_*}\bigg|_{z_*=0} = \mu a r f''(0),$$
$$\tau_r(z_* = h) = \mu\frac{\partial u_*}{\partial z_*}\bigg|_{z_*=h} = \mu a r\, f''(1). \tag{2.8}$$

3. The stretching–shrinking problem

When the upper plate stretches and the lower plate shrinks corresponding to $a > 0$, we vary σ over some range $\sigma \le 0$. We begin with analyses for the $R=0$ (Stokes) and the asymptotic large-R solutions. It should be immediately pointed out that an exact solution found by Aristov et al. [19] exists for the special case $\sigma = -1$ where the upper plate stretches at rate a and the lower plate shrinks at the same rate. The solution along with shear stresses and the pressure gradient parameter are given as

$$f(\eta) = \eta(1 + \eta), \quad f''(0) = f''(1) = 2, \quad \beta = -R. \tag{3.1}$$

We now proceed to determine the Stokes ($R=0$) solution and the large-R asymptotic behavior of solutions.

3.1. The Stokes solution

For $R=0$ we have the leading-order equation and boundary-conditions:

$$f''' - \beta_0 = 0, \quad f(0) = 0, \quad f'(0) = \sigma, \quad f(1) = 0, \quad f'(1) = 1 \tag{3.2}$$

the solution of which is given by

$$f(\eta) = \sigma\eta - (1 + 2\sigma)\eta^2 + (\sigma + 1)\eta^3, \quad f'(\eta)$$
$$= \sigma - 2(1 + 2\sigma)\eta + 3(\sigma + 1)\eta^2. \tag{3.3}$$

Thus the parameters measuring the radial shear stresses at the lower and upper walls and the pressure gradient parameter calculated from Eq. (2.7) for the Stokes solution are given by

$$f''(0) = -2 - 4\sigma, \quad f''(1) = 4 + 2\sigma, \quad \beta = 6 + 6\sigma \quad (R = 0). \tag{3.4}$$

3.2. Large-R asymptotics

It becomes clear from the integrations that the large Reynolds number asymptotics for the pressure gradient and the shear stress at the lower wall satisfy

$$\beta \sim -\sigma^2 R, \quad f''(0) \sim -2\sigma \quad (R \to \infty) \tag{3.5}$$

and these values are achieved at relatively small values of R on the order of 10^2. We use this information to extract the asymptotic nature of the upper wall shear stress in the analysis given below.

A coordinate $\zeta = 1 - \eta$ is first introduced giving the boundary-value problem:

$$\dddot{f} - R(2f\ddot{f} - \dot{f}^2) + \beta = 0, \tag{3.6a}$$

$$f(0) = 0, \quad \dot{f}(0) = -1, \quad f(1) = 0, \quad \dot{f}(1) = -\sigma \tag{3.6b}$$

where an overdot represents differentiation with respect to ζ. Now the dependent and independent variables are scaled with R in the following manner:

$$\xi = \sqrt{R}\zeta, \quad F(\xi) = \sqrt{R}\,f(\zeta) \tag{3.7}$$

and we substitute the asymptotic form $\beta \sim -\sigma^2 R$ to obtain the upper wall boundary-layer equation

$$F''' - 2FF'' + F'^2 - \sigma^2 = 0 \tag{3.8a}$$

where now a prime denotes differentiation with respect to ξ. The transformed boundary conditions are

$$F(0) = 0, \quad F'(0) = -1, \quad F(\sqrt{R}) = 0, \quad F'(\sqrt{R}) = -\sigma. \tag{3.8b}$$

In these boundary conditions, $R \to \infty$ so the $\xi \to \infty$ and the conditions outside the boundary layer are either $F'(\infty) = -\sigma$ or