International Journal of Non-Linear Mechanics 77 (2015) 248–253

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

General and approximate solutions of the boundary layer equation for radial flow of incompressible liquid

Nikolay A. Kudryashov*, Mark B. Kochanov

Department of Applied Mathematics, National Research Nuclear University "MEPhI", 31 Kashirskoe Shosse, 115409 Moscow, Russian Federation

A R T I C L E I N F O

Article history:
Received 2 April 2015
Received in revised form
2 September 2015
Accepted 2 September 2015
Available online 11 September 2015

Keywords:
Radial flow
Boundary layer equation
General solution
Quasi-exact solution

A B S T R A C T

Non-linear partial differential equation for description of a laminar, incompressible, Newtonian radial flow in half-subspace is derived taking into account free slipping at a border. The boundary layer approximation is used and the equation with one free parameter for a steam function is obtained. The general solution for one value of this parameter is found and analysed. Some approximate solutions of the special form are constructed.

1. Introduction

Prandtl's boundary layer theory was applied to the two-dimensional round steady laminar flow by Schlichting [1] and Bickley [2]. They studied a free two-dimensional jet emerging from a thin long hole into a fluid at rest and solved analytically resulting ordinary differential equation using self-similar variables. Later the application of the boundary layer theory to laminar jets was fully discussed in monographs [3,4].

Mason [5] derived a group invariant solution for a steady two-dimensional jet by considering a linear combination of the Lie point symmetries of Prandtl's boundary layer equations for jet. This solution has the form, assumed by Schlichting [1]. Schlichting also derived a conservation law for the differential equation for a stream function. Application of conservation laws as a systematic way to derive conserved quantities for jet is presented in [6]. Authors considered two-dimensional and axisymmetric jets with some types of boundary conditions. They applied the multiplier approach to a system for velocity components and a third-order partial differential equation for the stream function. The partial Lagrangian method is used to obtain same results for free jet in [7]. The steady two-dimensional boundary-layer equations in the flat and axisymmetric case were studied in [8]. Authors show that many proposed in many works new methods of reduction are indeed invariant solutions under the action of non-classical symmetries. Also they show that Mises transformation can be considered as Bäcklund transformation related to a non-classical symmetry. The non-classical symmetries of boundary layer equations for two-dimensional and axisymmetric flows have been considered also in [9]. In work [10] exact solutions of the equations of a stationary laminar boundary layer are reviewed. A general transformation of the three-dimensional boundary layer equations is presented in an arbitrary orthogonal curvilinear coordinate system. Extension of the Crocco transformation is used to investigate unsteady boundary layer equations in [11]. The review of equation describing the unsteady axisymmetric boundary layer on a body of revolution with arbitrary shape is presented in [12]. A number of new exact solutions involving two to five arbitrary functions are found.

In this work we derive the equation for radial flow of incompressible liquid in half-subspace over flat horizontal bottom. A free slipping boundary condition is used. We assume that the flow layer is thin, so we use the boundary layer approximation. Under this assumption, equation with a free parameter for a steam function is obtained. The general self-similar solution of this equation is obtained at particular conditions and three types of flows are given. Quasi-exact solutions are found to extend possible values of parameters in equation at which solution exists.

The outline of this manuscript is the following. In Section 2 we derive Prandtl's boundary layer equation for axial flow. Section 3 is devoted to finding of the general solution of equation at some

* Corresponding author.
E-mail address: nakudr@gmail.com (N.A. Kudryashov).

http://dx.doi.org/10.1016/j.ijnonlinmec.2015.09.001
0020-7462/

conditions. Some analysis of solution is performed. The quasi-exact solution at other conditions is found in Section 3.

2. Problem formulation

Let us consider the Navier–Stokes equations for axisymmetric flow when tangential velocity is equal to zero. We use the approximation of the boundary layer theory with a constant pressure field. In this case the Navier–Stokes equation for radial component of velocity and continuity equation in polar variables has the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{u}{x}\right)+\frac{\partial^2 u}{\partial y^2}\right) \tag{2.1}$$

$$\frac{\partial(xu)}{\partial x}+\frac{\partial(xv)}{\partial y}=0. \tag{2.2}$$

We assume that linear size of problem along x is L and the y-variation of the radial velocity is U. The laminar boundary layer is thin in comparison with L and has the size $\delta L(\delta \ll 1)$. From continuity equation (2.2) and condition at the bottom border

$$v|_{y=0}=0 \tag{2.3}$$

we get that $v\sim\delta U$. Using these assumptions, the parameter ν can be expressed via the Reynolds number

$$\nu=\frac{\mu}{\rho}=\frac{UL}{\mathrm{Re}}.$$

One can see that all members on the left-hand side of Eq. (2.1) has the order U^2/L. To keep the terms on the right-hand side of equation the Reynolds number must has the order $1/\delta^2$. The first two members in brackets on the right-hand side of considered equation are much smaller and we can neglect them. As a result we have

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2.4}$$

$$\frac{\partial(xu)}{\partial x}+\frac{\partial(xv)}{\partial y}=0. \tag{2.5}$$

We seek for a solution of system (2.4), (2.5) using the stream function:

$$u=\frac{1}{x}\frac{\partial\psi}{\partial y},\quad v=-\frac{1}{x}\frac{\partial\psi}{\partial x}. \tag{2.6}$$

After substituting (2.6) into Eqs. (2.4) and (2.5), the system is reduced to equation on $\psi(x,y)$ known as Prandtl's boundary layer equation for a radial flow

$$\frac{1}{x}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{1}{x^2}\left(\frac{\partial\psi}{\partial y}\right)^2-\frac{1}{x}\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\nu\frac{\partial^3\psi}{\partial y^3}=0, \tag{2.7}$$

where $\nu>0$ is the kinematic viscosity. It can be shown that Eq. (2.7) admits a dilation group of transformation, and we can search for a self-similar solution of Eq. (2.7) using new variables

$$\psi(x,y)=x^{2-\beta}H(z),\quad z=\frac{y}{x^\beta}. \tag{2.8}$$

Taking into account (2.8), Eq. (2.7) can be written in the form

$$\nu\frac{d^3H}{dz^3}+(2-\beta)H\frac{d^2H}{dz^2}+(2\beta-1)\left(\frac{dH}{dz}\right)^2=0. \tag{2.9}$$

This equation can be reduced to the well-known Chazy equations (see, for example, [13–15]) at two values of parameter β. At $\beta=1$ using the following transformations:

$$z=Lz',\quad H=2L\nu H'$$

we have Chazy-II equation

$$H'''=2HH''+2H'^2.$$

Applying

$$z=Lz',\quad H=\frac{L\nu}{3}H'$$

we have Chazy-III equation

$$H'''=2HH''-3H'^2.$$

The first of these equations can be easily integrated. The second one has Painlevé property and it is the simplest example of an ordinary differential equation whose solution has a movable natural boundary.

3. General solution of Eq. (2.9) at $\beta=1$

In the case of $\beta=1$ Eq. (2.9) takes the form

$$\nu\frac{d^3H}{dz^3}+H\frac{d^2H}{dz^2}+\left(\frac{dH}{dz}\right)^2=0$$

and can be integrated twice. We have the Riccati equation in the form [15]

$$\frac{dH}{dz}+\frac{1}{2\nu}H^2=C_1z+C_2, \tag{3.1}$$

where C_1 and C_2 are integration constants. Eq. (3.1) can be obtained by using exponential non-local symmetries [16,17]. This equation is invariant under transformations

$$(z,H,C_1)\mapsto(-z,-H,-C_1).$$

Without loss of generality we can assume that $z\ge 0$, i.e. $y\ge 0$. We have to consider two different cases: $C_1=0$ and $C_1\ne 0$.

In the first case ($C_1=0$) we have an equation for tanh-function and its real-valued solution has the form [18]

$$H(z)=\sqrt{2\nu C_2}\tanh\left(\sqrt{\frac{C_2}{2\nu}}z+C_3\right). \tag{3.2}$$

This solution can be obtained by simplest equation method. For example, Q-function method [19–21] gives the solution in the present form.

In the second case ($C_1\ne 0$) after changing of variables

$$(z,H)\mapsto\left(-z-\frac{C_2}{C_1},-2\nu H\right) \tag{3.3}$$

Eq. (3.1) can be written as

$$\frac{dH}{dz}+H^2+\frac{C_1}{2\nu}z=0.$$

It can be linearized by the transformation

$$H(z)\equiv\frac{y'(z)}{y(z)}.$$

As a result we have

$$y''+\frac{C_1}{2\nu}yz=0.$$

Rescaling the independent variable by formula

$$z\mapsto-\left(\frac{2\nu}{C_1}\right)^{1/3}z \tag{3.4}$$

we obtain the equation for the Airy function in the form

$$y''-yz=0. \tag{3.5}$$

It is interesting to note that Airy equation appears in solving Chazy-XXI equation in the special case [14]. Solution of Eq. (3.5)