International Journal of Mechanical Sciences 83 (2014) 91–95

Contents lists available at ScienceDirect

International Journal of Mechanical Sciences

journal homepage: www.elsevier.com/locate/ijmecsci

Ultimate load theorems for rigid plastic solids with internal degrees of freedom and their application in continual lattice shells

Vladimir A. Grachev, Yuriy S. Neustadt *

Samara State Architectural and Building University, 194 Molodogvardeyskaya Str., Samara 443001, Russia

ARTICLE INFO

Article history:
Received 13 September 2013
Accepted 27 March 2014
Available online 2 April 2014

Keywords:
Rigid plastic solids
Cartan hedron
Constitutive equations
Ultimate load
Shape memory
Rotating shells

ABSTRACT

This paper studies solids with internal degrees of freedom using the method of Cartan moving hedron. Strain compatibility conditions are derived in the form of structure equations for manifolds. Constitutive relations are reviewed and ultimate load theorems are proved for rigid plastic solids with internal degrees of freedom. It is demonstrated how the above theorems can be applied in behavior analysis of rigid plastic continual shells of shape memory materials. The ultimate loads are estimated for rotating shells under external forces and in case of shape recovery from heating.

1. Introduction

This paper consists of two parts. The first section covers ultimate load theorems [1–3] for solids with internal degrees of freedom. Then these theorems are extended to deformable bodies where internal degrees of freedom are represented by plastic hinges of shape memory materials.

In the second section of the paper general considerations are applied to the behavioral analysis of axially-symmetric continual shells compiled of thin plates that are connected with rigid plastic hinges.

2. General theory

Let us review a three-dimensional space (manifold) where points are designated by vector $\vec{r}$ in the fixed reference system. Each point P would be associated with an orthogonal trihedral while the letters $\vec{e}_1, \vec{e}_2, \vec{e}_3$ would mean the unit vectors of its axes. Let the space be filled with solids and the points $(\vec{r}+d\vec{r})$ close to $\vec{r}$ shall be defined by the new trihedral with the unit vectors of the axes that are different from the earlier introduced $d\vec{e}_1, d\vec{e}_2, d\vec{e}_3$. Decomposition of the vector differentials along the axes of the original trihedral looks as follows [4]

$$d\vec{r} = \omega^i \vec{e}_i, \quad d\vec{e}_i = \omega_{ij}\vec{e}_j \tag{1}$$

Here ω^i, ω_{ij} are differential forms that are linear against differentials of initial variables α_i. The domain of parameter variation is designated as $D = (A_1 < \alpha_1, \alpha_2, \alpha_3 < A_2)$.

The forms ω^i, ω_{ij} are not independent because of the relations

$$(\vec{e}_1)^2 = (\vec{e}_2)^2 = (\vec{e}_3)^2 = 1$$
$$\vec{e}_1\vec{e}_2 = \vec{e}_2\vec{e}_3 = \vec{e}_1\vec{e}_3 = 0$$

Six forms $\omega^1, \omega^2, \omega^3, \omega_{12}, \omega_{23}, \omega_{13}$ satisfy the equations of the structure

$$d\omega^i = [\omega^i_k \omega^k], \quad d\omega_{ij} = [\omega_{ik}\omega_{kj}], \quad \omega^i_k = -\omega_{ik} \tag{2}$$

The square brackets in formulas (2) mean exterior product of the forms while $d\omega_{ij}$ is an exterior differential of the form. Relations of the structure (2) are continuity equations of the deformable medium, the points of which can be attributed to movements $\vec{u} = \vec{r}_1 - \vec{r}$, where $\vec{r}_1$ is a location of the point P after deformation in the fixed reference system.

In fact, if there is a solution to the system (2), there is also a family of rectangular trihedrals so that the forms ω^i, ω_{ij} cause continuous manifold defined by vectors $\vec{r}$ and $\vec{r}_1$. Consequently, in order for the Eq. (2) to describe continuous movements of solids, they must contain as least six functions (parameters). But the system (1) includes nine functions (components of the vector $\vec{u}$ derivates) and it accepts various solutions depending on the property of the solids. Within the mechanics of deformable bodies, additional dependences are obtained based on the motion (equilibrium) equations and constituting equations between stresses and strains. In other words, a symmetric tensor of stresses σ_{ij} (six independent components) is introduced and connected by means

* Corresponding author.
E-mail address: neustadt99@mail.ru (Y.S. Neustadt).

http://dx.doi.org/10.1016/j.ijmecsci.2014.03.030
0020-7403/

of the constituting equations to the vector $\overrightarrow{u}$. Then taking into account the equilibrium (motion) conditions, a solvable system of nine equations is deduced.

But using derivatives of the vector $\overrightarrow{u}$ as variables is not an only option. Keeping in mind application to rigid plastic bodies, we assume that the forms ω^i, ω_{ij} depend on n internal parameters φ_i $(i=1,2...n)$, that have m dependencies like

$$\Phi_l(\varphi_i, \varphi_{i},_t) = 0, \quad l = 1, 2...m, \quad \varphi_{i},_t(\alpha_i) \in L_1(D),$$

$$\varphi_i = \int_0^t \varphi_{i},_t(\alpha_i, s)ds \tag{3}$$

The inferior index t after the comma means derivative with respect to time of the abstract function in the normed space $L_1(D)$. Therefore the number of independent functions φ_i equals to $p=n-m$. It turns out that if $p \geq 6$, the system of Eq. (2) determines the motion of the solid.

If $p<6$, the solid can move only as an absolutely rigid body.

Let us review the most important case when $p=6$, then the continuum forms a mechanism, which geometry is defined by the shape of the boundary ∂D within the domain D, while this mechanism moves without any bulk forces. We will study slow motions at constant temperature. In other words, we will replace the displacement vector $\overrightarrow{u}$ with the vector of velocities $\overrightarrow{v} = \overrightarrow{u},_t$ and will keep designations $\overrightarrow{e}_i$ for the unit vectors in the manifold of the velocities.

We will supplement (2) with equations following the principle of virtual power [5], that we will embrace the following way: there exist functions M_i (α_j)that for any virtual functions φ_i (α_j)satisfying (3), the identical Eq. (4) are met

$$\int_D M_i\, \varphi_{i,t}\, dV = \int_D F_i v_i\, dV + \int_{\partial D} P_i v_i\, dS \tag{4}$$

The letter D shall designate the domain filled with the solid, while ∂D is its boundary surface. The expression dV means an element of volume in point P and dS is an element of the boundary surface area where the stress vector P_i is defined. The values M_i represent internal stresses corresponding to virtual velocities $\varphi_{i,t}$. The letters F_i, P_i designate vectors of the bulk and surface forces. Since the parameters φ_i (α_j) are not independent, the identical Eq. (4) generates $n-m$ equations of equilibrium. If the connection between the real values of M_i and φ_i is known for the solid, the joint solution of (3) and (4) allows determining the stress-strain condition. Henceforth, we will analyze the rigid plastic solid, for which the correlation between M_i and φ_i look as follows:

$$\varphi_{i},_t = \lambda \frac{\partial f}{\partial M_i} \tag{5}$$

if $f=k$, $\partial f/\partial t = 0$, *then* $\lambda > 0$; *if* $f<k$, *or* $f=k$, $\partial f/\partial t < 0$, *then* $\lambda = 0$.

The function of n variables can be specified as $f(M_i)$, assuming that in case of $f(M_i)=k$ (here k is a constant), the function intercepts a convex domain D^T with the boundary S^T in the n-dimensional space (Fig. 1). In the ideal rigid plastic problem $f(M_i)$ is a homogeneous function of first degree, i.e. the below equality is fulfilled for any number c.

$$f(cM_i) = cf(M_i) \tag{6}$$

To be definite let us take

$$f = (\tfrac{1}{2} a_i^2 M_i^2)^{1/2} \tag{7}$$

The book [6] demonstrates that the relations (1)–(5) are equal to the problem of the functional minimum

$$L(\varphi_{i,t}) = \int_D \Pi(\varphi_{i,t})dV + \int_D F_i v_i\, dV + \int_{\partial D} P_i v_i\, dS \tag{8}$$

Here the dissipative potential $\Pi(\varphi_{i},_i)$ is Young's transform of the function

$$f^*(M_i) = 0, \text{ if } f \leq k \text{ and } f^*(M_i) = \infty, \text{ if } f > k$$

Thus,

$$\Pi(\varphi_{i},_t) = \sup_{M_i}(\varphi_{i},_t M_i - f^*(M_i)) \tag{9}$$

The variables M_i are selected from the manifold of continuous functions bounded above and $\varphi_{i},_t$ is a space of measures within the volume V. Repeating the arguments of the cited book, we will obtain the following results on the movement of the solids in question (ultimate load theorem):

Let $P_i=0$ and the load F_i is changing proportionally to the parameter p

$$F_i = p(t)F_{i0} \tag{10}$$

In these conditions there exists a number $p=p^*$ that there is both a minimum of the functional $L(\varphi_{i},_t)$ and such a solution to the system when the condition (5) is met.

The function $\lambda(\alpha_i)$ is defined accurate to the constant. Mechanically, the latter means that if $p=p^*$, the unconfined flow of the medium occurs.

The stresses in the points where plastic flow is observed are located on the surface $f=(\frac{1}{2}a_i^2M_i^2)^{1/2}=k$, and the following equality takes place

$$W = \int_D \Pi(\varphi_{i},_t)dV = -\int_D F_i v_i\, dV = -A \tag{11}$$

Here W is a rate of internal energy change and A is a power of external forces.

The presented results admit transfer to shape memory materials [7,8]. The plastic flow has the following special feature for these materials. If there is one loading surface at "room" temperature T_0

$$f = (\tfrac{1}{2} a_i^2 M_i^2)^{1/2} = k, \tag{12}$$

when the temperature is raised $T_0 < T_1 < ...T_i$ a new (inner) surface appears (Fig. 2) that is similar to the outer surface with the coefficient of similarity ε_i, so that $\varepsilon_1 > \varepsilon_2 > ...\varepsilon_i$. The domain bounded with the loading surfaces S^{T_i} and $S_\varepsilon^{T_i}$ shall be designated as $D_\varepsilon^{T_i}$. There exists a temperature, when the surface $S_\varepsilon^{T_i}$ collapses. This temperature T_ε is called shape restoration temperature.

Let us make sure that when the solid was strained at the "room" temperature to φ_i^0 under external loading, it would get

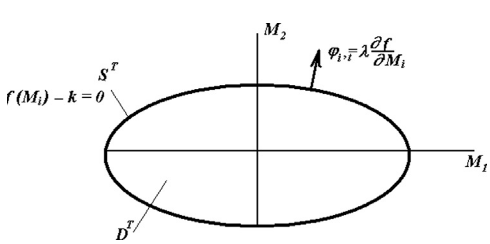

Fig. 1. Loading surface of rigid plastic materials.

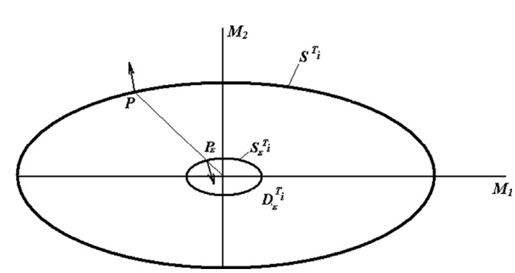

Fig. 2. Loading surface of shape memory materials.