International Journal of Non-Linear Mechanics 73 (2015) 64–68

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Stability of triangular libration points in a planar restricted elliptic three body problem in cases of double resonances

Olga Kholostova

Moscow Aviation Institute (National Research University), Volokolamskoe Shosse 4, Moscow 125993, Russia

ARTICLE INFO

Article history:
Received 25 February 2014
Received in revised form
16 August 2014
Accepted 3 November 2014
Available online 13 November 2014

Keywords:
Three-body problem
Triangle libration points
Stability
Double resonance

ABSTRACT

Stability of triangular libration points in a planar restricted elliptic three-body problem is investigated for two sets of parameters corresponding to the cases of double resonances. These positions are shown to be formally stable in the case of the fourth-order double resonance. When a weak third-order and a strong fourth-order resonances occur in the system, any motion of an approximate autonomous system is found to be bounded if initial conditions are sufficiently close to the unperturbed motion. Boundedness domain estimate is obtained.

1. Introduction

Stability investigation of the triangular libration points in a three-body problem is a classical problem of celestial mechanics. These constant positions were discovered by G.-L. Lagrange [1] for the Newtonian gravitational field and by P.-S. Laplace [2] for an arbitrary power law of attraction. Later necessary conditions for stability of the triangular libration points were obtained for the circular restricted three-body problem [3], the unrestricted three-body problem [4,5], and an arbitrary law of attraction [4–6]. In the paper [5] conditions for stability (in the first approximation) were also found for the elliptic problem. Non-linear stability analysis for the case of the circular problem is carried out in the papers [7,8]. Detailed non-linear stability analysis for the triangular libration points in the restricted circular and elliptic three-body problem for planar and spatial cases is given in the monograph [9]; here one can find bibliography on the subject. Stability of the triangular libration points in the case of critical mass ratio is investigated in the paper [10]. For the unrestricted three-body problem, a non-linear stability analysis of Lagrangian and Laplacian solutions is carried out in the papers [11–13].

2. Problem statement

Suppose two massive bodies S and J move in a field of a mutual gravitational attraction around their center of mass in elliptic orbits. The third body P of much less mass moves in the orbital plane of S and J affected by their attraction. Thus we consider a planar elliptic restricted three-body problem. The Hamiltonian function of the problem has a form [9]

$$H=\frac{1}{2}(p_\xi^2+p_\eta^2)+p_\xi\eta-p_\eta\xi+\frac{e\cos\nu}{2(1+e\cos\nu)}(\xi^2+\eta^2)-\frac{W}{1+e\cos\nu},$$
$$W=\frac{1-\mu}{r_1}+\frac{\mu}{r_2},\quad \mu=\frac{m_2}{m_1+m_2},$$
$$r_1=\sqrt{(\xi+\mu)^2+\eta^2},\quad r_2=\sqrt{(\xi+\mu-1)^2+\eta^2}. \tag{1}$$

Here ξ,η and p_ξ,p_η denote Nehvil's variables and corresponding momenta, e and ν eccentricity and true anomaly, and m_1 and m_2 ($m_1>m_2$) masses of the bodies S and J.

The system with Hamiltonian (1) has a particular solution

$$\xi_0=\frac{1-2\mu}{2},\quad \eta_0=\frac{\sqrt{3}}{2},\quad p_{\xi_0}=-\frac{\sqrt{3}}{2},\quad p_{\eta_0}=\frac{1-2\mu}{2}. \tag{2}$$

This solution corresponds to one of two constant configurations called the triangular libration point when the bodies lie (in the rotating coordinate system) at vertices of an equilateral triangle.

Introduce perturbations of quantities (2) by formulas

$$\xi=\xi_0+q_1,\quad \eta=\eta_0+q_2,\quad p_\xi=p_{\xi_0}+p_1,\quad p_\eta=p_{\eta_0}+p_2. \tag{3}$$

The Hamiltonian of the perturbed motion has the form [9]

$$H=H_2+H_3+H_4+\cdots,$$
$$H_2=\frac{1}{2}(p_1^2+p_2^2)+p_1q_2-q_1p_2+\frac{e\cos\nu(q_1^2+q_2^2)}{2(1+e\cos\nu)}+\frac{q_1^2-8kq_1q_2-5q_2^2}{8(1+e\cos\nu)},$$
$$H_3=\frac{\sqrt{3}}{1+e\cos\nu}\left(-\frac{7k}{36}q_1^3+\frac{3}{16}q_1^2q_2+\frac{11k}{12}q_1q_2^2+\frac{3}{16}q_2^3\right),$$

E-mail address: kholostova_o@mail.ru

http://dx.doi.org/10.1016/j.ijnonlinmec.2014.11.005
0020-7462/

$$H_4 = \frac{1}{1+e\cos\nu}\left(\frac{37}{128}q_1^4 + \frac{25k}{24}q_1^3q_2 - \frac{123}{64}q_1^2q_2^2 - \frac{15k}{8}q_1q_2^3 - \frac{3}{128}q_2^4\right), \quad (4)$$

where $k = 3\sqrt{3}(1-2\mu)/4$. The dots denote the set of terms of the fifth and higher orders in q_j, p_j $(j=1,2)$.

Fig. 1 shows a domain (in the μ, e plane) of stability in the linear approximation of solution (2) [9]. In this domain the characteristic exponents $\pm i\lambda_j$ $(j=1, 2)$ of a linearized system of equations of the perturbed motion are purely imaginary. The lower boundary $e=0$ corresponds to the case of a circular problem, for which

$$0 < \mu < \mu^* = \frac{9-\sqrt{69}}{18} = 0.0385208.$$

The upper and right boundaries of the domain are defined by the equations $2\lambda_2 = -1$ and $\lambda_1 + \lambda_2 = 0$ respectively.

There are seven resonance curves of the third and fourth orders defined by the equations

(1) $3\lambda_1 + \lambda_2 = 2$, (2) $3\lambda_1 - \lambda_2 = 3$, (3) $2\lambda_1 + \lambda_2 = 1$,

(4) $\lambda_1 + 3\lambda_2 = -1$, (5) $\lambda_1 - 2\lambda_2 = 2$, (6) $4\lambda_1 = 3$, (7) $3\lambda_2 = -2$.

Here the number of the equation coincides with the number of the corresponding curve in Fig. 1.

On curves 3 and 7 of strong third-order resonances (indicated by dotted lines in Fig. 1) the solution in question is unstable. On curves 1, 4 and 6 of strong fourth-order resonances the intervals of stability in the fourth approximation are indicated by solid lines, whereas the instability parts by dotted lines. On curves 2 and 5 of weak resonances of the third and fourth orders respectively (dash-dot lines) the solution is formally stable when there are no other resonances.

The domain contains two points of double resonance. At the point A ($e_1 = 0.165115$, $\mu_1 = 0.0393625$) the double fourth-order resonance occurs for which $\lambda_1 + 3\lambda_2 = -1$, $4\lambda_1 = 3$. This point belongs to the interval of stability in the fourth approximation of both the resonance curves. The point B ($e_2 = 0.1218928$, $\mu_2 = 0.03871614$) is the intersection point of the weak third-order resonance curve $\lambda_1 - 2\lambda_2 = 2$ and the fourth-order resonance curve $4\lambda_1 = 3$ for which the solution considered is stable in the fourth approximation.

We will show that the triangle vibration point is formally stable at the point A. We will also show that all motions of an approximate autonomous system corresponding to the point B are bounded if they begin in a sufficiently small neighborhood of the libration point. Boundedness domain estimate will be obtained.

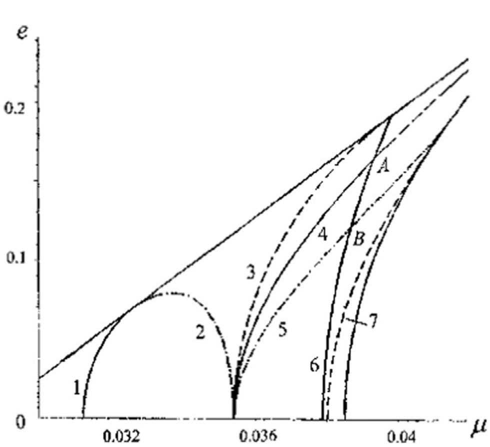

Fig. 1. Domain of linear stability and resonance curves.

3. Double fourth-order resonance investigation

We put in Hamiltonian (4) $e = e_1$, $\mu = \mu_1$, $k = k_1$ and make a canonical transformation $q_j, p_j \to q_j', p_j'$ $(j=1, 2)$ reducing it to a normal form in terms up to the fourth order inclusive in the perturbations in accordance with the available resonances. Normalization procedure is accomplished by means of the symbolic computation system Maple. It is not presented here due to the inconvenience. In the "polar" coordinates φ_j, r_j $(q_j' = \sqrt{2r_j}\sin\varphi_j$, $p_j' = \sqrt{2r_j}\cos\varphi_j)$ the normalized Hamiltonian has the form

$$H = \lambda_1 r_1 + \lambda_2 r_2 + c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 + \alpha r_1^2\cos(4\varphi_1 - 3t + 4\varphi_1^*)$$
$$+ \beta r_1^{1/2}r_2^{3/2}\cos[(\varphi_1 + 3\varphi_2) + t + (\varphi_1^* + 3\varphi_2^*)] + O_5, \quad (5)$$

where $\lambda_1 = 3/4$, $\lambda_2 = -7/12$. The calculations show that

$$c_{20} = 58.717491, \quad c_{11} = 45.544761, \quad c_{02} = 141.644459;$$
$$\alpha = 38.835299, \quad \beta = 162.754565, \quad \varphi_1^* = 1.171487, \quad \varphi_2^* = -1.811563. \quad (6)$$

We make the univalent canonical change of variables $\varphi_j, r_j \to \Phi_j$, R_j $(j=1,2)$ in (5) with the generating function

$$S = (\varphi_1 + \varphi_1^* - \lambda_1 t)R_1 + (\varphi_2 + \varphi_2^* - \lambda_2 t)R_2. \quad (7)$$

It is defined by the formulas

$$R_j = r_j, \quad \Phi_j = \varphi_j + \varphi_j^* - \lambda_j t \quad (j=1,2). \quad (8)$$

As a result the linear in r_j terms in the Hamiltonian vanish, time t disappears from the resonance terms, and the Hamiltonian has a form

$$H_1 = (c_{20} + \alpha\cos 4\Phi_1)R_1^2 + c_{11}R_1R_2 + c_{02}R_2^2 + \beta R_1^{1/2}R_2^{3/2}\cos(\Phi_1 + 3\Phi_2) + O_5. \quad (9)$$

First we consider the approximate Hamiltonian H_{10} truncating the last term on the right-hand side of equality (9). It can be written in the form

$$H_{10} = R_2^2 f_1(u), \quad f_1(u) = (c_{20} + \alpha n_1)u^4 + c_{11}u^2 + \beta n_2 u + c_{02}. \quad (10)$$

Here we introduce the notations

$$R_1/R_2 = u^2, \quad n_1 = \cos 4\Phi_1, \quad n_2 = \cos(\Phi_1 + 3\Phi_2) \quad (|n_j| \le 1,\ j=1,2).$$

Taking into consideration of numerical values (6), we can easily show that the fourth-order polynomial $f_1(u)$ takes strictly positive values for any $u > 0$. Indeed, the leading coefficient of the function f_1 and the second derivative $f_1'' = 12(c_{20} + \alpha n_1)u^2 + 2c_{11}$ are positive. Therefore, the first derivative $f_1' = 4(c_{20} + \alpha n_1)u^3 + 2c_{11}u + \beta n_2$ is a monotonous function with a unique root. In the case $n_2 > 0$ this root is negative. Hence, for any $u > 0$ we have $f_1' > 0$, i.e. the function f_1 is monotone increasing and $f_1(u) > f_1(0) = c_{02} > 0$.

In the case $n_2 < 0$ the root of the function f_1' is positive, and the graph of the function f_1 is a parabola which can intersect the abscissa axis at two points or can lie above this axis. In the boundary case of a double root when the abscissa axis is tangent to the parabola at the point of minimum two conditions $f_1 = f_1' = 0$ satisfy simultaneously. In this case the resultant

$$Rs = -(c_{20} + \alpha n_1)^2 g,$$
$$g = 27\beta^4(c_{20} + \alpha n_1)n_2^4 - 4\beta^2 c_{11}[36c_{02}(c_{20} + \alpha n_1) - c_{11}^2]n_2^2$$
$$-16c_{02}[4c_{02}(c_{20} + \alpha n_1)^2 - c_{11}^2]^2. \quad (11)$$

of the functions f_1 and f_1' vanishes.

Notice that the quadratic in n_2^2 trinomial in (11) always has two real roots of opposite signs. The positive one, however, is greater than 1, since the inequality

$$g|_{|n_2|=1} = -256c_{02}^3\alpha^2 n_1^2 + \alpha[9\beta^2(3\beta^2 - 16c_{11}c_{02})$$
$$+128c_{02}^2(c_{11}^2 - 4c_{20}c_{02})]n_1$$
$$+27c_{20}\beta^4 + 4c_{11}(c_{11}^2 - 36c_{20}c_{02})\beta^2$$
$$-16c_{02}(c_{11}^2 - 4c_{20}c_{02})^2 < 0$$

is satisfied for $|n_1| \le 1$ and values (6) under consideration.