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On the influence of radial displacements and bending strains on the large deflections of impulsively loaded circular plates

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This work considers the validity of the assumption that the influence of radial displacements and bending strains can be ignored when constructing closed form analytical solutions for the large displacement response of impulsively loaded circular plates. Three successive energy based analyses are presented. The first analysis considers only membrane strains, but includes radial displacements in addition to transverse displacements. It is shown that the inclusion of radial displacements alters the plastic strain distribution but does not alter the integral of the total energy dissipated due to plastic strain, and consequently, does not affect the final central deflection estimate. This result provides a rigorous justification for the assumption that radial displacements can be ignored. However, though neglectable, the radial displacements are not negligible and must be included in order to obtain realistic strain distributions. Thereafter, a model is presented that considers the interaction of bending and membrane strains. It is shown that at large displacements the membrane strains suppress the effect of bending strains. This result supports the assumption that bending strains can be ignored at large deflections. Furthermore, it provides an estimate for the displacement range in which the interaction between bending and membrane effects is significant and should not be ignored. Lastly, the non-monotonic nature of the strain history is considered and shown to have a small effect.

1. Introduction

The body of literature relating to the impulsive loading of thin circular plates is extensive [1–4] and contains numerous analytical solutions covering various loading and response conditions. Theoretical models describing plate response characteristics are generally categorized according to the assumptions upon which they are based. For example, a common approach is to assume an ideal impulsive load [5–8], *i.e.* the load is considered to be evenly distributed across a plate with a duration that is negligible in comparison with the plate response duration, and consequently, the plate is viewed as having an instantaneous uniform initial velocity.

An important parameter is the magnitude of the final central deflection relative to the plate thickness. For small displacements, *i.e.* central deflections of up to half a plate thickness, bending strains are assumed to dominate [9]. Under these circumstances, assuming that the plate material is rigid-perfectly plastic and obeys the Tresca yield condition with an associated flow rule, it is possible to obtain an exact solution for the equations of motion [10,11]. An example of this is the solution reported by Wang [5], which captures important transient features, such as a plastic hinge that originates at the plate boundary and travels radially inwards.

For large displacements, *i.e.* central deflections that are several times greater than a plate thickness, membrane strains that arise from transverse, *i.e.* out of plane, displacements are assumed to dominate [9], and the plate is typically treated as a rigid-perfectly plastic membrane. A variety of analytical solutions to this problem have been published [6,7,12–18].

Taylor [12] outlined an analytical approach which considers a plastic hinge travelling radially inwards at a constant speed and predicts a conical final plate shape. Similar models have been presented by Hudson [13], Frederick [14] and Wierzbicki and Nurick [18].

Duffey and Key [6,15] used an energy approach where the final deflection is calculated by equating the initial kinetic energy to the total energy dissipated through plastic deformation. This approach requires the displacement profile of the plate to be assumed *a priori*. Furthermore, it is implicitly assumed that the strain history is monotonic, *i.e.* the total dissipated energy depends only on the final plate shape and is not affected by transient features, such as a travelling hinge.

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Symonds and Weirzbicki [7] used the mode approximation technique developed by Martin and Symonds [19], and argued that qualitatively meaningful results could be obtained by considering only the governing equations of the final phase of motion. This aspect is similar to the energy approach in that transient plate profiles were not considered. Their analysis resulted in a final plate shape in the form of a zeroth order Bessel function of the first kind.

A common feature of all the above-mentioned solutions is the assumption that the effect of radial, *i.e.* in plane, displacements can be ignored. In the small displacement range, this assumption is reasonable and conforms to well established plate bending theory [1]. In the large displacement range, the assumption has been motivated by analogy, numerical results and experimental data. Taylor [12] argued that the dynamic behaviour of a perfectly plastic thin plate is analogous to a vibrating constant tension membrane, the solution of which does not require the radial displacement to be considered. A different approach was presented by Duffey and Key [6,15] who used the results of numerical simulations to argue that radial displacements are negligible, while Jones [9] cited the lack of radial displacements in the experimental data of Griffith and Vanzant [20].

However, the assumption has also been called into question. In particular, Nurick *et al.* [21] noted that neglecting radial displacements results in a strain distribution that is essentially the inverse of that which is obtained experimentally. Furthermore, they presented a numerical mode approximation model, which included radial displacements, and produced a strain distribution that matched the experimental trend, which implies that radial displacements are not negligible.

Another common feature of the aforementioned solutions is that the interaction of bending and membrane strains is not considered. This approach is not valid at intermediate displacements, *i.e.* final central deflections in the range of a plate thickness, for which a limited number of analyses have been published.

Jones [9] appears to be the first to incorporate both bending moments and membrane forces in a solution for an impulsively loaded circular plate. The plate response was divided into two phases. Phase I was assumed to be bending dominated and resulted in a solution that is identical to that of Wang [5] up to the point where the travelling hinge reaches the plate centre. Thereafter, Phase II was assumed to be membrane dominated and used the final phase I displacement and velocity distributions as initial conditions for phase II. The analysis is limited in that it is not applicable to problems where the travelling hinge persists into the large deflection regime. Furthermore, while it incorporates both membrane and bending effects, it does not consider their combined effect, *i.e.* the bending and membrane effects are separate and uncoupled.

In subsequent work, Jones [22,1] presented solutions for the impulsive loading of annular and circular plates that incorporated combined, but uncoupled, bending and membrane effects. These analyses are based on a yield condition originally proposed by Hodge [23], which is a simplification of that presented by Onat and Prager [24], who considered two bending moment and two membrane force components that display non-linear interactions [24,23]. The simplification proposed Hodge maintains force–force and moment–moment interactions, but neglects interactions between forces and moments, *i.e.* they occur simultaneously but do not interact. The motivation for the simplification is that the general equations of motion based on the interactive yield condition have proved to be mathematically intractable for all but the simplest problems [9]. Hence, no exact closed form solutions using the interactive yield condition are known, although Yu and Chen [17] have presented a detailed approximate model incorporating interactive yielding.

Wen [8] used a similar simplified yield condition with an energy method incorporating a parabolic assumed displacement plate profile. Unlike previous energy solutions for circular plates, Wen considered the dissipation of energy due to simultaneous bending and membrane strains, but in a summative sense without coupling.

The models discussed above, and their common assumptions, appear to be representative of the literature. To the authors' knowledge, there is no published closed form analytical solution for the impulsive loading of thin circular plates that treats bending and membrane strains in a coupled sense. Furthermore, the effect of radial displacements, and their interactions with bending deformations, do not appear to be accounted for.

The purpose of this paper is threefold: Firstly, to present a theoretical justification for the assumption that radial displacements can be ignored when considering the maximum deflection of an impulsively loaded thin circular plate in the large deflection regime. Secondly, to present a model that incorporates coupled interaction between bending and membrane strains, and use it to estimate the displacement range for which the interaction should not be ignored. Lastly, to consider the non-monotonic nature of the strain history and assess its effect.

2. Radial displacement distribution

While absent in analytical models of impulsively loaded circular plates, deformation theories that include radial deflections have been included in the analysis of quasi-statically loaded membranes. Taylor [12] presented an approximate radial deflection solution that will form the basis of an analysis presented in this paper and will, therefore, be considered in detail.

Taylor considered the radial equilibrium of a perfectly plastic static membrane undergoing large deflection. The angle between the membrane and the horizontal plane was assumed to be small and, hence, the cosine of the angle was treated as unity, leading to a radial equilibrium equation of form

$$r\frac{d\sigma_r}{dr}+\sigma_r-\sigma_\theta=0 \tag{1}$$

where σ_r and σ_θ are the membrane stress is the radial and circumferential directions, respectively.

Assuming that the membrane thickness does not change appreciably, the perfect plasticity assumption implies a constant flow stress which satisfies Eq. (1), since the variation of stress in the radial direction vanishes and the radial and circumferential stresses are equal at every point. Consequently, Taylor argued that the Mises yield criterion and associated flow rule state that the radial and circumferential strains will also be equal [12].

For an axisymmetric membrane that experiences radial displacements in addition to large transverse deflections, the strain distribution, ignoring higher order terms, is given by

$$\varepsilon_r=\frac{1}{2}\left(\frac{dw}{dr}\right)^2+\frac{du}{dr}\quad\text{and}\quad\varepsilon_\theta=\frac{u}{r} \tag{2}$$

where w and u are the displacement distributions in the transverse and radial directions, respectively, while ε_r and ε_θ are the strain distributions in the radial and circumferential directions, respectively [12,25,9].

Taylor assumed a parabolic transverse displacement profile, which has the form

$$w=\tilde{w}\left[1-\left(\frac{r}{R_o}\right)^2\right] \tag{3}$$

where $\tilde{w}$ is the central deflection and R_o is the outer radius.

Taylor used a parabolic profile because it is the exact solution for a constant tension membrane subjected to uniform static load, but it has also been applied to impulsively loaded plates