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Fully non-linear wave models in fiber-reinforced anisotropic incompressible hyperelastic solids

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ABSTRACT

Many composite materials, including biological tissues, are modeled as non-linear elastic materials reinforced with elastic fibers. In the current paper, the full set of dynamic equations for finite deformations of incompressible hyperelastic solids containing a single fiber family are considered. Finite-amplitude wave propagation ansätze compatible with the incompressibility condition are employed for a generic fiber family orientation. Corresponding non-linear and linear wave equations are derived. It is shown that for a certain class of constitutive relations, the fiber contribution vanishes when the displacement is independent of the fiber direction.

Point symmetries of the derived wave models are classified with respect to the material parameters and the angle between the fibers and the wave propagation direction. For planar shear waves in materials with a strong fiber contribution, a special wave propagation direction is found for which the non-linear wave equations admit an additional symmetry group. Examples of exact time-dependent solutions are provided in several physical situations, including the evolution of pre-strained configurations and traveling waves.

1. Introduction

The majority of open problems in the theory of elasticity and related areas and applications stem from the essential non-linearity of the governing equations. Starting from the landmark work of Haramard and Rivlin [19,34], a large number of theoretical results have been obtained in the field of non-linear elastostatics, especially in the case of incompressible materials. The study of non-linear wave propagation in the elastodynamic setting has received significant attention. Substantial work in the theory of non-linear elastic waves in pre-stressed solid bodies relies on the linearization of the equations of non-linear elastodynamics with respect to small perturbations superimposed on a state of homogeneous or inhomogeneous finite strain. The propagation of acoustic waves within finitely deformed elastic materials was first considered in [45], based on the superimposition of small-amplitude oscillations on finite initial homogeneous deformations. This pioneering work was then followed by other contributions, including [3,43,44,46]. It was more specifically shown in those works that pure longitudinal and pure transverse waves can only propagate in the so-called specific directions, depending on material symmetries [7]. A formulation of elastic energy density for an isotropic medium involving small-but-finite amplitude waves, relying on certain approximations, has been developed in [20].

In contrast to the approximate treatment of non-linear models based on the incremental analysis (linear approximation) of the field equations, the more general situation of finite amplitude waves leads to fully non-linear models (e.g., [30]). Fewer theoretical results are available for such models. Fundamental work relevant to this contribution includes that of Carroll [8], where finite-amplitude incompressible elastic waves in non-linear isotropic materials were studied, and linearly polarized motions in one and two dimensions, and circularly polarized transverse waves were considered. The topic of finite amplitude waves in finitely deformed solids is extensively discussed in [38] and references therein. In the last twenty years, much attention has been devoted to classes of materials with specific constitutive relations, such as the neo-Hookean and Mooney–Rivlin hyperelastic materials. Exact solutions of various non-linear elasticity models for specific types of motions have been derived in a number of papers, including [15,36].

The analysis of non-linear wave propagation in pre-strained or pre-stressed elastic solid bodies is of particular interest in a number of mechanical and physical areas including geophysics, electronics, earthquake engineering, composite materials, ultrasonic non-destructive analysis of soft biological tissues. Initial stress and/or strain frequently occurs during the manufacturing and assembly of structural elements, such as composite materials, fiber reinforced solids (including dry textiles). Moreover, non-zero

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stress and strain are often naturally present in soft biological tissues such as veins, arteries, muscles, ligaments and tendons (see, e.g., [22]); for instance, skin is in a state of natural tension.

The propagation of non-linear waves in many materials, including soft tissues, may be further complicated by anisotropy effects [42]. The anisotropy of the deformation pattern can be induced, for example, by an initial state of finite deformation, onto which further motions are superimposed, or by the existence of a fibrous microstructure which has preferred directions. Examples of soft biological tissues which display this type of microstructure are arteries (see [1,22,23]) and skin (e.g., [2,35]). The analysis of non-linear waves in soft tissues consisting of a soft matrix reinforced by fibers requires the theory of fiber-reinforced isotropic hyperelastic materials. Various constitutive models of fiber-reinforced materials have been introduced, including models for one and two families of fibers. For such models, the determination of the elastic constants is more involved than in the isotropic case due to the presence of additional anisotropy parameters. Some of these models are reviewed below.

A number of important theoretical results have been obtained for fiber-reinforced material models. A possibility of singular behavior of transversely isotropic (single fiber family) incompressible materials under inhomogeneous shear was discovered in [28]. It was shown that depending on the reinforcement strength, the static stress field may become discontinuous, which is associated with fiber kinking and the loss of ellipticity of the field equations. In [17], equilibrium states of incompressible materials with two parallel families of fibers under shear deformations were considered; it was shown that strain singularities can develop when mechanical differences between the two fibers families are sufficiently large. Practical implications of such singular behaviors in biological and other models have been discussed.

On the side of applications, the topic of non-linear wave propagation in anisotropic media has received considerable attention due to its implication in imaging techniques aiming at accessing mechanical properties or internal visualization of organs (e.g., [48]). A number of imaging techniques have been developed to characterize tissue stiffness *in vivo* by measuring the shear wave speed propagation [37,40,41].

Invariance properties of the governing equations are an important aspect of the theoretical understanding of linear and non-linear physical models. The Lie point symmetry framework and related methods provide systematic ways of study of invariance properties of differential equations (DE) with respect to continuous and discrete symmetry groups. Local symmetries have been widely used to obtain exact solutions of DEs, as symmetry-invariant solutions, or through mappings of known solutions into new ones. Many techniques for exact solution of ordinary and partial differential equations (ODE, PDE), including superposition principles, integral transforms, existence of separated solutions, reduction of order for ODEs, construction of Green's functions, existence of traveling wave solutions, etc., are directly related to symmetry properties of the equations under study. In particular, the invariance under space and time translations validates the traveling wave solution ansatz, which is essential for soliton equations, such as the Korteweg–de Vries equation, the non-linear Schrödinger equation, and the Sine–Gordon equation. Most of the wave equations of physical interest in fact admit larger symmetry groups. For variational PDE systems, local conservation laws and variational symmetries are equivalent through Noether's theorem. This is generally not so for non-variational PDE systems (see, e.g., [4,5,6,31]). An extensive study of the relationship between symmetries and conservation laws appears in [31] and references therein. We note that a number of important equations of mathematical physics, including the Khokhlov–Zabolotskaya and the integrable Kadomtsev–Petviashvili and models, arise in non-linear elasticity context (e.g., [13]).

Symmetries of the two-dimensional Ciarlet–Mooney–Rivlin model of compressible isotropic hyperelastic materials have been analyzed in [9]. A special value of the traveling wave speed has been found for which the non-linear Ciarlet–Mooney–Rivlin equations admit an additional infinite set of point symmetries. A family of essentially two-dimensional traveling wave solutions has been derived for that case. An overview of related recent results based on the application of symmetry methods to elastodynamics equations can also be found in [9].

The current contribution is concerned with the theoretical study of fully non-linear wave propagation models in fiber-reinforced hyperelastic incompressible materials, with a specific goal of finding closed-form time-dependent exact solutions in two spatial dimensions. In the present work, we restrict our attention to the case of a single family of fibers. Wave propagation ansätze compatible with the incompressibility condition, for a general fiber family orientation, are used. The work is based on the Lie point symmetry classification of the governing equations. It is planned to address the more general situation of two families of fibers, which is necessary, for example, for an adequate description of biological tissues [22], in subsequent work.

An outline of the present contribution is as follows. The equations of motion for incompressible hyperelastic solids reinforced with a single family of fibers are reviewed in Section 2. Several constitutive relations are discussed. In our work, we restrict our attention to incompressible Mooney–Rivlin-type materials.

In Section 3, motions with a special orientation with respect to the fiber family are considered. In particular, in the general fully non-linear setting, a theorem is proven stating that if the displacement is independent of the fiber direction, then the fiber-dependent invariant I_4 is a constant. It follows that in models where a constitutive relation for the fiber-reinforced material involves a fiber contribution only through I_4, motions with the indicated displacements will not "feel" the presence of fibers. Examples of such motions are considered in the following sections. A similar statement in a totally different context of the incremental analysis and linearized equations has been made in [29]. There, it is noted that "…the shear wave solution involving only deformation in the plane of isotropy is not affected by anisotropic term in the constitutive equation". In the current paper, we show that even in the full non-linear setting, motions with displacements independent of the fiber direction are indeed not affected by the anisotropic terms if the constitutive relation only involves the invariant I_4. The effect of anisotropy will still be present if the model involves other fiber-dependent invariants, for example, I_5 (see Section 2).

Fully non-linear anti-plane shear motions, with displacements orthogonal to a plane, are considered in Section 4, for an arbitrary fiber family orientation. Displacements dependent on one and two spatial variables are analyzed. Lie point symmetries are computed in one- and two-dimensional cases. For the two-dimensional case, the displacement satisfies a non-linear wave equation with a differential constraint (cf. [21,25,47]). The loss of hyperbolicity in the model is discussed; a sufficient condition of hyperbolicity is derived. A single non-linear wave equation is derived for one-dimensional transverse wave (s-wave) propagation; its sample numerical solutions are obtained. It is proven that the one-dimensional model admits an extra symmetry for a special fiber orientation. The condition for the existence of the extra symmetry corresponds to the boundary of the domain of model parameters in which the loss of hyperbolicity of the PDE may occur. The additional symmetry is used to construct an exact symmetry-invariant solution describing the degeneration of the parabolic shear into a simple linear shear as $t \to \infty$.

Another situation where displacements are orthogonal to an axis is studied in Section 5, for the general and specific fiber family orientations. Here displacements in the two transverse directions and the hydrostatic pressure satisfy a $1+1$-dimensional non-linear