International Journal of Non-Linear Mechanics 71 (2015) 132–151

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Asymptotic profiles for the third grade fluids equations

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ARTICLE INFO

Article history:
Received 6 October 2013
Accepted 21 March 2014
Available online 29 December 2014

Keywords:
Fluid mechanics
Third grade fluids
Asymptotic expansion

ABSTRACT

We study the long time behaviour of the solutions of the third grade fluids equations in dimension 2. Introducing scaled variables and performing several energy estimates in weighted Sobolev spaces, we describe the first order of an asymptotic expansion of these solutions. It shows in particular that, under smallness assumptions on the data, the solutions of the third grade fluids equations converge to self-similar solutions of the heat equations, which can be computed explicitly from the data.

1. Introduction

The study of the behaviour of the non-Newtonian fluids is a significant topic of research in mathematics, but also in physics or biology. Indeed, these fluids, the behaviour of which cannot be described with the classical Navier–Stokes equations, are found everywhere in the nature. For examples, blood, wet sand or certain kind of oils used in industry are non-Newtonian fluids. In this paper, we investigate the behaviour of a particular class of non-Newtonian fluids that is the third grade fluids, which are a particular case to the Rivlin–Ericksen fluids (see [29,30]). The constitutive law of such fluids is defined through the Rivlin–Ericksen tensors, given recursively by

$$A_1 = \nabla u + (\nabla u)^t,$$
$$A_k = \partial_t A_{k-1} + u \cdot \nabla A_{k-1} + (\nabla u)^t A_{k-1} + A_{k-1} \nabla u,$$

where u is a divergence free vector field of $\mathbb{R}^2$ or $\mathbb{R}^3$ which represents the velocity of the fluid. The most famous example of a Rivlin–Ericksen fluid is the class of the Newtonian fluids, which are given through the stress tensor

$$\sigma = -pI + \nu A_1,$$

where $\nu > 0$ is the kinematic viscosity and p is the pressure of the fluid. Introduced into the equations of conservation of momentum, this stress tensor leads to the well known Navier–Stokes equations.

In this paper, we consider a larger class of fluids, for which the stress tensor is not linear in the Rivlin–Ericksen tensors, but a polynomial function of degree 3. As introduced by Fosdick and Rajagopal in [13], the stress tensor that we consider is defined by

$$\sigma = -pI + \nu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta |A_1|^2 A_1,$$

where $\nu > 0$ is the kinematic viscosity, p is the pressure, $\alpha_1 > 0$, $\alpha_2 \in \mathbb{R}$ and $\beta \geq 0$.

We assume in this paper that the density of the fluid is constant in space and time and equals 1. Actually, the value of the density is not significant, since we can replace the parameters ν, α_1, α_2 and β by dividing them by the density. Introduced into the equations of conservation of momentum, the tensor σ leads to the system

$$\begin{aligned} &\partial_t (u - \alpha_1 \Delta u) - \nu \Delta u + \operatorname{curl}(u - \alpha_1 \Delta u) \wedge u \\ &\quad - (\alpha_1 + \alpha_2)(A \cdot \Delta u + 2\operatorname{div}(LL^t)) - \beta \operatorname{div}\left(|A|^2 A\right) + \nabla p = 0, \\ &\operatorname{div} u = 0, \\ &u_{|t=0} = u_0, \end{aligned} \tag{1.1}$$

where $L = \nabla u$, $A(u) = \nabla u + (\nabla u)^t$ and $\wedge$ denotes the classical vectorial product of $\mathbb{R}^3$. For matrices $A, B \in \mathcal{M}_d(\mathbb{R})$, we define $A : B = \sum_{i,j=1}^d A_{i,j} B_{i,j}$ and $|A|^2 = A : A$. If the space dimension is 2, we use the convention $u = (u_1, u_2, 0)$ and $\operatorname{curl} u = (0, 0, \partial_1 u_2 - \partial_2 u_1)$. Notice also that if $\alpha_1 + \alpha_2 = 0$ and $\beta = 0$, we recover the equations of motion of second grade fluids, which are another class of non-Newtonian fluids, introduced earlier by Dunn and Fosdick in 1974 (see [10,15] or [9]). If in addition $\alpha_1 = 0$, then one recovers the classical Navier–Stokes equations.

The system of Eq. (1.1) has been studied in various cases, on bounded domains of $\mathbb{R}^d$, $d=2$, 3 or in the whole space $\mathbb{R}^d$ (see [1–5,22,26]). On a bounded domain Ω of $\mathbb{R}^d$ with Dirichlet boundary conditions, Amrouche and Cioranescu have shown the existence of local solutions to (1.1) when the initial data belong to the Sobolev space $H^3(\Omega)^d$ (see [1]). In addition, these solutions are unique. For this study, the authors have assumed the restriction

$$|\alpha_1 + \alpha_2| \leq (24 \nu \beta)^{1/2},$$

which is justified by thermodynamics considerations. The proof of their result is obtained via a Galerkin method with functions belonging to the eigenspaces of the operator curl $(I - \alpha_1 \Delta)$. In

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http://dx.doi.org/10.1016/j.ijnonlinmec.2014.11.018
0020-7462/

dimension 3, a slightly different method has been applied by Bresch and Lemoine, who used Schauder's fixed point theorem to extend the result of [1] to the case of initial data belonging to the Sobolev spaces $W^{2,r}(\Omega)^3$, with $r>3$. They have shown in [3] the local existence of unique solutions of (1.1) in the space $C^0([0,T], W^{2,r}(\Omega)^3)$, where $T>0$. In addition, if the data are small enough in the space $W^{2,r}(\Omega)^3$, the solutions are global in time. Notice also that the existence of such solutions holds without restrictions on the parameters of the system (1.1).

In the case of third grade fluids filling the whole space $\mathbb{R}^d$, $d=2,3$, Busuioc and Iftimie have established the existence of global solutions with initial data belonging to $H^2(\mathbb{R}^d)^d$, without restrictions on the parameters or on the size of the data (see [4]). In this study, the authors used a Friedrichs scheme and performed a priori estimates in H^2 which allow to show the existence of solutions of (1.1) in the space $L^\infty_{loc}(\mathbb{R}^+, H^2(\mathbb{R}^d)^d)$. Besides, these solutions are unique if $d=2$. Later, Paicu has extended the results of [4] to the case of initial data belonging to $H^1(\mathbb{R}^d)^d$, assuming additional restrictions on the parameters of the equation; the uniqueness is not known in this space (see [26]). The method that he used is slightly different from the one used in [4]. Indeed, although Paicu also considered a Friedrichs scheme, the convergence of the approximate solutions to a solution of (1.1) is done via a monotonicity method. Notice that Theorem 1.1 of this paper shows the existence of solutions of the equations of third grade fluids on $\mathbb{R}^2$ for initial data in weighted Sobolev spaces (see Section 3).

In what follows, we consider a third grade fluid filling the whole space $\mathbb{R}^2$. Actually, the equations that we consider are not exactly the system (1.1) but the one satisfied by $w=\operatorname{curl} u=\partial_1 u_2-\partial_2 u_1$. In dimension 2, the vorticity equations of the third garde fluids are given by

$$\begin{aligned}
&\partial_t(w-\alpha_1\Delta w)-\nu\Delta w+u\cdot\nabla(w-\alpha_1\Delta w)\\
&\qquad-\beta\operatorname{div}\left(|A|^2\nabla w\right)-\beta\operatorname{div}\left(\nabla\left(|A|^2\right)\wedge A\right)=0,\\
&\quad\operatorname{div} u=0,\\
&\quad w_{|t=0}=w_0=\operatorname{curl} u_0.
\end{aligned}\tag{1.2}$$

Notice that the parameter α_2 does no longer appear in (1.2) and thus does not play any role in the study of these equations. Indeed, due to the divergence free property of u, a short computation shows that $\operatorname{curl}\left(A\cdot\Delta u+2\operatorname{div}(LL^t)\right)=0$, or equivalently there exists q such that $A\cdot\Delta u+2\operatorname{div}(LL^t)=\nabla q$. This phenomenon is very particular to the dimension 2 and does not occur in dimension 3. Notice also that the previous system is autonomous in w. Indeed, the vector field u depends on w and can be recovered from w via the Biot–Savart law, which is a way to get a divergence free vector field such that $\operatorname{curl} u=w$. The motivation for considering the vorticity equations instead of the equations of motion comes from the fact that, due to spectral reasons, we have to study the behaviour of the solutions of (1.2) in weighted Lebesgue spaces. Indeed, in what follows, we will consider scaled variables, which make appear a differential operator whose essential spectrum can be "pushed to the left" by taking a convenient weighted Lebesgue space. We will see that the rate of convergence of the solutions of (1.2) is linked to the spectrum of this operator. Unfortunately, the weighted Lebesgue spaces are not suitable for the equations of motions and are not preserved by the system (1.1). Anyway, one can obtain the asymptotic profiles of the solutions of the equations of motion (1.1) from the study of the asymptotic behaviour of the solutions of the vorticity equations (see Corollary 1.1 below). We also emphasize that the system (1.2) allows to consider solutions whose velocity fields are not bounded in L^2.

In this paper, we establish the existence and uniqueness of solutions of (1.2) in weighted Sobolev spaces, but the main aim is the study of the asymptotic behaviour of these solutions when t goes to infinity. More precisely, we want to describe the first order asymptotic profiles of the solutions of (1.2). We consider a fluid of third grade which fills $\mathbb{R}^2$ without forcing term applied to it. In this case, as it is expected, the solutions of (1.2) tend to 0 as t goes to infinity. Our motivation is to show that these solutions behave like those of the Navier–Stokes equations. In our case, we will show that the solutions of (1.2) behave asymptotically like solutions of the heat equations, up to a constant that we can compute from the initial data. The methods that we use in the present paper are based on scaled variables and energy estimates in several functions spaces. This work is inspired by several older results obtained for other fluid mechanics equations. The first and second order asymptotic profiles have been described for the Navier–Stokes equations in dimensions 2 and 3 by Gallay and Wayne (see [18–21]). In dimension 2, they have shown in [18,20] that the first order asymptotic profiles of the Navier–Stokes equations are given up to a constant by a smooth Gaussian function called the Oseen vortex sheet. More precisely, for a solution w of the vorticity Navier–Stokes equations (that is the system 1.2 with $\alpha_1=\beta=0$), for every $2\le p\le+\infty$, the following property holds:

$$\left\|w(t)-\frac{\int_{\mathbb{R}^2}w_0(x)\,dx}{t}G\left(\frac{\cdot}{\sqrt{t}}\right)\right\|_{L^p}=\mathcal{O}(t^{-3/2+1/p})\quad\text{when } t\to+\infty,$$

where G is the Oseen vortex sheet

$$G(x)=\frac{1}{4\pi}e^{-|x|^2/4}.\tag{1.3}$$

The methods that they used in [18] are very different from the ones that we develop in this paper. Although they also considered scaled variables, the convergence to the asymptotic profiles is not obtained through energy estimates. Indeed, using dynamical systems arguments, they established the existence of a finite-dimensional manifold which is locally invariant by the semiflow associated to the Navier–Stokes equations. Then, they showed that, under restrictions on the size of the data, the solutions of the Navier–Stokes equations behave asymptotically like solutions on this invariant manifold. The description of the asymptotic profiles is thus obtained by the description of the dynamics of the Navier–Stokes equations on the invariant manifold. Later, the smallness assumption on the data has been removed (see [20]). In [24], Jaffal-Mourtada describes the first order asymptotics of second grade fluids, under smallness assumptions on the initial data in weighted Sobolev spaces. She has shown that the solutions of the second grade fluids equations converge also to the Oseen vortex sheet. In this paper, we apply the methods used by Jaffal-Mourtada, namely scaled variables and energy estimates. According to these results, one can say that the fluids of second grade behave asymptotically like Newtonian fluids. In this paper, we show that, under the same smallness assumptions on the initial data, the same behaviour occurs for the third grade fluids equations. We emphasize that the rate of convergence that we obtain is better than the one obtained in [24]. Actually, we show that we can choose the rate of convergence as close as wanted to the optimal one, assuming that the initial data are small enough. Since second grade fluids are a particular case of third grade fluids, we establish an improvement of the rate obtained in [24]. Actually, the main difference between third and second grade fluids equations in dimension 2 is the presence of the additional term $\beta\operatorname{div}\left(|A|^2A\right)$ in the third grade fluids equations. Sometimes, this term helps to obtain global estimates, like in [4] or [26], but introduces additional difficulties when one looks for estimates in H^3 or in more regular Sobolev spaces (see [1,2] or [5]). Here, we have to establish estimates in weighted Sobolev spaces with H^2 regularity for the vorticity w, which is harder than doing estimates in H^3 for u.

We next introduce scaled variables. In order to simplify the notations, we assume that $\nu=1$. Let $T>1$ be a positive constant which is introduced in order to avoid restrictions on the size of the parameter α_1 and which will be made more precise later. We consider the solution w of (1.2) and define W and U such that $\operatorname{curl} U=W$ through the change of variables $X=x/\sqrt{t+T}$ and $\tau=\log(t+T)$.