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A Timoshenko beam element based on the modified couple stress theory

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ABSTRACT

Since the classical continuum theory is neither able to evaluate the accurate stiffness nor able to justify the size-dependency of micro-scale structures, the non-classical continuum theories such as the modified couple stress theory have been developed. In this paper, a new comprehensive Timoshenko beam element has been developed on the basis of the modified couple stress theory. The shape functions of the new element are derived by solving the governing equations of modified couple stress Timoshenko beams. Subsequently, the mass and stiffness matrices are developed using energy approach and Hamilton's principle. The formulations of the modified couple stress Euler–Bernoulli beam element and also classical Timoshenko and Euler–Bernoulli beam elements can be recovered from the original formulations of the new Timoshenko beam element. By two examples, it is indicated that how the new beam element can be applied to deal with the real-case problems. The static deflection of a short microbeam and pull-in voltage of an electrostatically actuated microcantilever made of silicon are evaluated by employing the new beam element and the results are compared to the experimental data as well as the classical FEM results. It is observed that the results of the new beam element are in good agreement with the experimental findings while the gap between the classical FEM and experimental results is notable.

1. Introduction

Micro/nano-scale mechanical components, such as micro/nano-beams, are the major building blocks of micro/nano electromechanical systems (MEMS/NEMS) [1,2] and atomic force microscopes (AFMs) [3,4]. Hence, investigating the mechanical behavior of such components has always been an important issue among the researchers. Due to complications existing in micro/nano-scale systems such as the presence of the complex forces like electrostatic, Casimir, Van Der Waals and capillary forces, complex geometry or some other issues like existence of the squeeze film damping, the exact analytical solutions may not be achieved for the behavior of the mechanical components in many cases; so, some approaches other than analytical one are required. One of the most popular approaches is the finite element method (FEM). The FEM is utilized by many researchers in order to investigate the mechanical behavior of micro/nano-scale systems. For example:

- Analysis of piezoelectric cantilever type beam actuators: Wu et al. [5].
- Study of the mechanical behavior of conducting polymer electromechanical actuators (CPEA): Metz et al. [6].
- Investigation of the static behavior and pull-in voltage of electrostatically actuated cantilever microswitches: Coutu et al. [7].
- Analysis of the mechanical response of microswitches with piezoelectric actuation: Cahpius et al. [8]
- Investigating the dynamic pull-in of an electrostatically actuated micro/nano-plate considering geometrical nonlinearities and fluid pressure by employing a nine-node plate element: Tajalli et al. [9].
- Modeling the MEMS subjected to electrostatic forces by developing a non-conforming element: Rochus et al. [10].

It is noted that all the above-mentioned works are based on the elements developed by the classical continuum theory.

The experimental observations have indicated that the classical continuum mechanics not only underestimates the stiffness of micro/nano-scale structures but also is incapable of justifying the size-dependency observed in these structures [11–13]; noted that the size-dependency is a peculiar phenomenon in which the normalized mechanical quantities of micro-scale structures that the classical continuum theory predicts to be independent of the structure size, significantly changes by the size. Hence, during past

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years, some non-classical continuum theories such as the couple stress theory have been emerged, developed, modified and employed to study the mechanical behavior of the micro-scale structures.

The couple stress theory has been introduced and developed by some researchers such as Koiter, Ejike, Mindlin and Tiersten in early 1960s [14–16]. In this theory beside the two classical material constants (i.e. the elastic modulus and Poison's ratio) additional material parameters are appearing which enable the theory to predict and model the size-dependency in micro-scale structures. Asghari et al. developed a Timoshenko beam model based on this theory to investigate the size-dependent static behavior of microbeams [17]. They concluded that the bending stiffness of the new model is greater than those evaluated based on the classical Timoshenko beam theory.

Yang et al. [18] performed a modification on the couple stress theory and presented the modified couple stress theory. They utilized the equilibrium equation of moments of couples in addition to two classical equilibrium equations i.e. the equilibrium equation of forces and moment of forces. Soon after that, this new theory became a popular non-classical theory. Many researchers utilized the modified couple stress theory to develop beam and plate models as well as investigate the size-dependent phenomena in microsystems. Some of these works on developing beam and plate models can be listed as below:

- Linear homogenous Euler–Bernoulli beam model by Park and Gao [19] and Kong et al. [20].
- Linear homogenous Timoshenko model by Ma et al. [21].
- Nonlinear homogenous Euler–Bernoulli beam model by Xia et al. [22] and Kahrobaiyan et al. [23].
- Nonlinear homogenous Timoshenko beam model by Asghari et al. [24]
- Linear functionally graded Euler–Bernoulli and Timoshenko beam models by Asghari et al. [25,26].
- Linear homogenous Kirchhoff plate model by Tsiatas [27].

In addition to developing beam and plate models, mechanical behavior of microsystems have also been investigated and analyzed based on the modified couple stress theory. Some of these works can be expressed as

- Investigating the vibration of fluid-conveying microtubes by Wang [28].
- Analyzing the buckling of micro-tubules by Fu and Zhang [29].
- Studying the dynamic characteristics of atomic force microscopes (AFMs) by Kahrobaiyan et al. [30].
- Investigating the size-dependent static behavior of electrostatically actuated microcantilevers and micro-bridges by Rahaeifard et al. [31,32].

Due to:

- vast applications micro-scale components such as microbeams in MEMS/NEMS,
- necessity of employing the non-classical continuum theories in order to capture the size-dependency and evaluate reliable stiffness for micro-scale structures,
- necessity of utilizing the FEM in MEMS/NEMS because of the complexities in these systems,

developing the structural finite elements based on the non-classical continuum theories seems to be essential. Recently, non-classical bar and Euler–Bernoulli beam elements are developed on the basis of the strain gradient elasticity, a non-classical continuum theory [33,34]. Since Timoshenko beam element is an important structural element that has been widely investigated in the literature [35–40], a modified couple stress Timoshenko beam element is developed in this paper. The new beam element is a comprehensive beam element that the formulations of the modified couple stress Euler–Bernoulli beam element as well as the classical Timoshenko and Euler–Bernoulli beam elements can be achieved by letting some parameter to zero in the original formulations. The shape functions of the new beam element are derived by directly solving the static equilibrium equations of modified couple stress Timoshenko beams. The stiffness and mass matrices are developed on the basis of the Hamilton's principle. Some examples are prepared to indicate that how the newly developed beam element can apply to the real-case problems and by comparing the results of the new beam element with the experimental data, it is indicated that the new beam element can successfully capture the size-dependency unlike the classical beam elements. In addition, it is observed that the results of the new beam element are in good agreement with the experimental results whereas the gap between the experimental and the classical FEM outcomes is considerable. The first example deals with the static deflection of a short microcantilever subjected to a concentrated force at its free end. In this example, the results of the modified couple stress and classical Timoshenko and Euler–Bernoulli beam elements are compared to the experimental data and it is observed that the new modified couple stress Timoshenko beam element has the best agreement with experimental findings while the gap between the classical FEM and experimental results is significant. In the second example, the static pull-in voltage of an electrostatically actuated microcantilever made of silicon is determined utilizing the new beam element and the present results are compared to the experimental and the classical FEM results. It is observed that the results of the new beam element are in good agreement with the experimental findings unlike the results of the classical beam element.

2. Preliminaries

Strain energy U of an elastic continuum occupying the volume of V modeled by the modified couple stress theory can be mentioned as [18]

$$U=\frac{1}{2}\int_V(\sigma_{ij}\varepsilon_{ij}+m_{ij}\chi_{ij})\,dV \quad (i,j=1,2,3), \tag{1}$$

where σ_{ij}, ε_{ij}, m_{ij} and χ_{ij} refer to the components of the classical stress and strain tensors, the symmetric part of the couple stress tensor and the symmetric part of the curvature (or rotation gradient) tensor respectively. The strain energy of a modified couple stress continuum consists of two parts: (1) a classical part, i.e. $(1/2)\sigma_{ij}\varepsilon_{ij}$ and (2) a non-classical part, i.e. $(1/2)m_{ij}\chi_{ij}$.

For an isotropic material, the strain tensor can be related to the stress tensor via Hooke's law as

$$\varepsilon_{ij}=\frac{1}{E}\left[(1+\nu)\sigma_{ij}-\nu\sigma_{kk}\delta_{ij}\right]=\frac{1}{2\mu}\left[\sigma_{ij}-\frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right], \tag{2}$$

where E and μ represent the elastic (Young's) modulus and the shear modulus respectively and ν stands for Poisson's ratio (noted that $E=2(1+\nu)\mu$). In addition, the couple stress tensor is related to the curvature tensor as [18]

$$m_{ij}=2l^2\mu\chi_{ij}, \tag{3}$$

in which l denotes the material length scale parameter, an additional material parameter enabling the theory to capture the size-dependency. The strain tensor can be expressed as the