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Closed-form solution for buckling analysis of thick functionally graded plates on elastic foundation

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ABSTRACT

Closed-form solution for buckling analysis of thick functionally graded plate resting on elastic foundation is presented using the third-order shear deformation theory. It is assumed that the plate rests on Pasternak foundation and its material properties vary through the plate thickness as a power function. By decoupling the governing equations, the neutral surface position for such plate is determined, and the third-order shear deformation theory based on exact neutral surface position is employed to derive the governing equations. Comparing with the middle surface based formulations, the neutral surface based formulations do not exhibit the stretching–bending coupling; hence, the values of buckling load can be obtained by eigenvalue analysis. Closed-form solutions are obtained for rectangular with different boundary conditions. Numerical results are presented and discussed for a wide range of plate and foundation parameters.

1. Introduction

Functionally graded materials are a class of composites that have continuous variation of material properties from one surface to another and thus eliminate the stress concentration found in laminated composites. Due to the variation of material properties through the thickness, the stretching–bending coupling exists in the functionally graded (FG) plate. This coupling produces transverse deflection and bending moments when plate is subjected to in-plane compressive loads. Hence, bifurcation-type buckling will not occur [1,2]. The conditions for bifurcation-type buckling to occur under the action of in-plane loads are examined by Aydogdu [3] and Naderi and Saidi [4,5] for FG plates, and by Leissa [6] and Singh et al. [7] for unsymmetrically laminated plates. It is observed that the bifurcation-type buckling occurs when the plate is fully clamped. For movable-edge plate, the bifurcation-type buckling occurs when the in-plane loads are applied at the neutral surface [5,8]. Although many researchers studied the buckling of FG plate using bifurcation-type analysis [9–14], they did not realize that the movable-edge FG plates exhibit a bifurcation-type buckling only if the applied load acts on the neutral surface. Therefore, the buckling analysis is presented herein for FG plate subjected to in-plane loads acting on the neutral surface.

The neutral surface of FG plate may not coincide with its middle surface due to the stretching–bending coupling. This coupling can be eliminated if the governing equations are derived based on the neutral surface [15,16]. Several studies on FG plates have been carried out using neutral surface based formulation. For example, Zhang and Zhou [17] used the classical plate theory (CPT) based on neutral surface to study the bending, buckling, and free vibration responses of FG plates. The CPT based on neutral surface was also adopted by Bodaghi and Saidi [18] to study the buckling of FG plates under nonlinearly varying in-plane loads resting on elastic foundation. Since the CPT ignores the transverse shear deformation effects, their reported results are limited to thin plates. To account for the transverse shear deformation effects, Prakash et al. [19] employed the first-order shear deformation theory (FSDT) based on neutral surface to investigate the influence of neutral surface position on the nonlinear stability behavior of FG plates. Recently, Singha et al. [20] employed the FSDT based on neutral surface to study the nonlinear behaviors FG plates under transversely distributed load. Although the FSDT provides sufficiently accurate result for moderately thick plate, it is not convenient to use due to requiring shear correction factor [21–37]. To avoid the use of shear correction factors, several higher-order shear deformation theories have been developed. Among them, the third-order shear deformation theory (TSDT) of Reddy [38] is the most widely used due to its accuracy and efficiency. Therefore, it is adopted herein for buckling analysis of FG plates on elastic foundation.

In this paper, the TSDT [38] is modified with respect to actual neutral surface for buckling analysis of FG plate on elastic

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foundation. It is assumed that the plate rests on Pasternak foundation and its material properties vary through the plate thickness as a power function. By decoupling middle surface based governing equations, the position of neutral surface is determined, and the governing equations based on neutral surface are derived. Comparing with the middle surface based formulations, the neutral surface based formulations do not exhibit the stretching–bending coupling, and consequently, the closed-form solutions of bifurcation buckling load can be obtained using eigenvalue analysis. The accuracy of the presented results is verified, and the effects of power law index, thickness ratio, aspect ratio, and foundation parameters on the buckling load of FG plates are discussed.

2. Governing equations based on middle surface

Based on the TSDT, the displacement field is assumed to be [38]

$$
\begin{aligned}
u_1(x,y,z) &= u(x,y) + z\phi_x(x,y) - \frac{4z^3}{3h^2}\left(\phi_x + \frac{\partial w}{\partial x}\right)\\
u_2(x,y,z) &= v(x,y) + z\phi_y(x,y) - \frac{4z^3}{3h^2}\left(\phi_y + \frac{\partial w}{\partial y}\right)\\
u_3(x,y,z) &= w(x,y)
\end{aligned}
\tag{1}
$$

where u and v are the in-plane displacements in the x and y directions, respectively; ϕ_x and ϕ_y are the rotation of the middle surface in the x and y directions, respectively; w is the transverse displacement; and h is the plate thickness. The linear strains can be obtained as

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} \varepsilon_x^1 \\ \varepsilon_y^1 \\ \gamma_{xy}^1 \end{Bmatrix} + f\begin{Bmatrix} \varepsilon_x^3 \\ \varepsilon_y^3 \\ \gamma_{xy}^3 \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = g\begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} \tag{2}
$$

where

$$
\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix},
$$

$$
\begin{Bmatrix} \varepsilon_x^3 \\ \varepsilon_y^3 \\ \gamma_{xy}^3 \end{Bmatrix} = \begin{Bmatrix} \partial\phi_x/\partial x + \partial^2 w/\partial x^2 \\ \partial\phi_y/\partial y + \partial^2 w/\partial y^2 \\ \partial\phi_x/\partial y + \partial\phi_y/\partial x + 2\partial^2 w/\partial x\partial y \end{Bmatrix}
$$

$$
\begin{Bmatrix} \varepsilon_x^1 \\ \varepsilon_y^1 \\ \gamma_{xy}^1 \end{Bmatrix} = \begin{Bmatrix} \partial\phi_x/\partial x \\ \partial\phi_y/\partial y \\ \partial\phi_x/\partial y + \partial\phi_y/\partial x \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} = \begin{Bmatrix} \phi_x + \partial w/\partial x \\ \phi_y + \partial w/\partial y \end{Bmatrix},
$$

$$
f = -\frac{4z^3}{3h^2}, \quad g = 1 - 4\left(\frac{z}{h}\right)^2 \tag{3}
$$

The stresses can be determined from the constitutive relations of a FG plate as

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \frac{E(z)}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{4}
$$

where ν is Poisson's ratio of FG plate assumed to be constant; $E(z)$ is Young's modulus of FG plate and is expressed according to a power law function as

$$
E(z) = E_m + (E_c - E_m)\left(\frac{1}{2} + \frac{z}{h}\right)^p \tag{5}
$$

where the subscripts m and c represent the metallic and ceramic constituents, respectively; and p is the power law index. The value of p equal to zero represents a fully ceramic plate, whereas infinite p indicates a fully metallic plate. Using the principle of minimum total potential energy, the governing equations are obtained as

$$
\delta u: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{6a}
$$

$$
\delta v: \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{6b}
$$

$$
\delta\phi_x: \quad \frac{\partial(M_x + P_x)}{\partial x} + \frac{\partial(M_{xy} + P_{xy})}{\partial y} - Q_x = 0 \tag{6c}
$$

$$
\delta\phi_y: \quad \frac{\partial(M_{xy} + P_{xy})}{\partial x} + \frac{\partial(M_y + P_y)}{\partial y} - Q_y = 0 \tag{6d}
$$

$$
\delta w: \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \left(\frac{\partial^2 P_x}{\partial x^2} + 2\frac{\partial^2 P_{xy}}{\partial x\partial y} + \frac{\partial^2 P_y}{\partial y^2}\right) - K_w w + K_s\nabla^2 w + \tilde{N} = 0 \tag{6e}
$$

where

$$
(N_i, M_i, P_i) = \int_{-h/2}^{h/2}(1, z, f)\sigma_i dz \quad (i = x, y, xy)
$$

and

$$
Q_i = \int_{-h/2}^{h/2} g\sigma_{iz} dz \quad (i = x, y) \tag{7}
$$

$$
\tilde{N} = N_x^0\frac{\partial^2 w}{\partial x^2} + N_y^0\frac{\partial^2 w}{\partial y^2} + 2N_{xy}^0\frac{\partial^2 w}{\partial x\partial y} \quad \text{and} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{8}
$$

where N_x^0, N_y^0, N_{xy}^0 are in-plane pre-buckling forces; K_w and K_s are the transverse and shear stiffness coefficients of the foundation, respectively.

By substituting Eqs. (2) into (4) and the subsequent results into Eq. (7), the stress resultants are obtained as

$$
\begin{Bmatrix} \{N\} \\ \{M\} \\ \{P\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [B^s] \\ [B] & [D] & [D^s] \\ [B^s] & [D^s] & [H^s] \end{bmatrix}\begin{Bmatrix} \{\varepsilon^0\} \\ \{\varepsilon^1\} \\ \{\varepsilon^3\} \end{Bmatrix} \quad \text{and} \quad \{Q\} = [A^s]\{\gamma^0\} \tag{9}
$$

where

$$
([A],[B],[B^s],[D],[D^s],[H^s]) = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}(A, B, B^s, D, D^s, H^s) \tag{10a}
$$

$$
(A, B, B^s, D, D^s, H^s) = \int_{-h/2}^{h/2}\frac{E(z)}{1-\nu^2}(1, z, f, z^2, zf, f^2)\, dz \tag{10b}
$$

$$
[A^s] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}A^s, \quad A^s = \int_{-h/2}^{h/2}\frac{E(z)}{1-\nu^2}g^2\, dz \tag{10c}
$$

3. Decoupling the governing equations

By substituting Eqs. (9) into (6), the governing equations can be rewritten as

$$
A\left(\frac{\partial\varphi_1}{\partial x} + \frac{1-\nu}{2}\frac{\partial\varphi_2}{\partial y}\right) + (B + B^s)\left(\frac{\partial\varphi_3}{\partial x} + \frac{1-\nu}{2}\frac{\partial\varphi_4}{\partial y}\right) + B^s\nabla^2\left(\frac{\partial w}{\partial x}\right) = 0 \tag{11a}
$$

$$
A\left(\frac{\partial\varphi_1}{\partial y} - \frac{1-\nu}{2}\frac{\partial\varphi_2}{\partial x}\right) + (B + B^s)\left(\frac{\partial\varphi_3}{\partial y} - \frac{1-\nu}{2}\frac{\partial\varphi_4}{\partial x}\right) + B^s\nabla^2\left(\frac{\partial w}{\partial y}\right) = 0 \tag{11b}
$$

$$
\begin{aligned}
&(B + B^s)\left(\frac{\partial\varphi_1}{\partial x} + \frac{1-\nu}{2}\frac{\partial\varphi_2}{\partial y}\right) + (D + 2D^s + H^s)\left(\frac{\partial\varphi_3}{\partial x} + \frac{1-\nu}{2}\frac{\partial\varphi_4}{\partial y}\right)\\
&\quad - A^s\left(\phi_x + \frac{\partial w}{\partial x}\right) + (D^s + H^s)\nabla^2\left(\frac{\partial w}{\partial x}\right) = 0
\end{aligned}
\tag{11c}
$$