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A new development and interpretation of the Navier–Stokes fluid which reveals why the "Stokes assumption" is inapt

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ABSTRACT

In this short paper I present an alternate way of describing fluids in general, and the Navier–Stokes fluid in particular, from a phenomenological point of view, that shows clearly that the putative assumption conjectured by Stokes is not a reasonable assumption. I also show that the procedure presented here is more suited for incorporating constraints such as that of incompressibility, as well as having other advantages. The approach also helps to pinpoint several serious errors in the justifications that are provided in classical texts for the development of the Navier–Stokes model. Finally, from the point of view of the role that causality plays in Newtonian mechanics, the approach suggested is the preferable approach.

1. Introduction

The Navier–Stokes model is unquestionably one of the most successful constitutive relations in mechanics. It has been so well studied and discussed that a paper that questions the methodology that has been used for arriving at it from the phenomenological point of view (Navier and Poisson used a molecular approach), and moreover avers that the classical approach does not reflect the philosophical underpinnings of Newtonian mechanics would, if not dismissed off hand, be the object of much derision and ridicule. Thus, it is with the greatest of inquietude and trepidation that I am espousing such a view and hoping that the reader would take an attitude that is ephectic, give me a fair hearing, and without condemning me unheard, pronounce judgment after careful rumination. Of course, one that dares to question conventional wisdom ought to be willing to put up with whatever condign punishment might be meted out, in case such questioning is misplaced.

The central argument of this paper is that a reassessment of the methodology for developing of the Navier–Stokes constitutive relation from the phenomenological point of view is necessary for a variety of reasons. The tenets of a successful theory usually go unquestioned; a fate of most successful theories seems to be their being misused and abused. The initial success of some theories could be due to happenstance or serendipity, the application of the theory being fortuitously directed to special circumstances wherein the inaptness of the theory is not pellucid. The Navier–Stokes theory on the other hand has been tested so thoroughly and for so long that it is unlikely that it's basic assumptions are fatally flawed. However, it is yet possible that by carefully reappraising the basic precepts and principles of the theory and refining them, one could attain a better understanding of the basis of the theory and thereby its possible generalizations. The analysis that is out carried out here bares some of the errors that are usually made in the interpretation and use of the Navier–Stokes theory, but more importantly it points to some inherent inadequacies of the theory as presented within the context of the traditional viewpoint. The assessment also clarifies some unsatisfactoriness with regard to the incorporation of constraints such as incompressibility that has persisted since their treatment by D'Alembert and Bernoulli (which was also followed by Lagrange [19]; Gauss [8] provided the correct methodology for dealing with constraints with regard to particle mechanics.). Had the new perspective presented herein been the way in which the theory had been developed, we would not have controversies such as those that surround the Stokes assumption[1]. The generalization of the methodology also points to a novel approach to the problem of Turbulence, the most important open problem in mechanics.

[1] Stokes [53] suggested that in a great many flows wherein the density of the fluid remains nearly constant, that is when the flow is nearly isochoric, one could make the assumption that $3\lambda+2\mu=0$, where λ and μ are material moduli that appear in the Navier–Stokes model (see Eq. (1.6) that follows); the assumption that $3\lambda+2\mu=0$, is referred to in fluid mechanics as the "Stokes assumption". This relationship however has been subsequently used indiscriminately to simplify the Navier–Stokes equation. Though all the experimental evidence that is available unequivocally contradict the assumption, there is yet no clear theoretical argument in place that puts paid to the assumption. A succinct and clear discussion of the controversy regarding the Stokes assumption can be found in a lengthy Appendix in Truesdell [59]. While the Appendix discusses various experiments that contradict Stokes assumption and the theoretical attempts at discrediting the same, there is no compelling theoretical argument that is provided that negates the Stokes assumption. This article is devoted to the development of such a theoretical basis.

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In this paper I provide an alternate modus operandi for obtaining the classical Navier–Stokes constitutive relation which I believe is more in harmony with the philosophical underpinnings of Newtonian mechanics. The molding of the theory not only allows for a simple and elegant treatment of constraints such as incompressibility, it is also much more in keeping with causality that is at the heart of Newtonian physics. Moreover, the expatiation of the general theory leads to a fresh and novel way of deriving the Navier–Stokes constitutive relation that resolves certain issues that are at the moment contentious and arguable. At the heart of the paper, is the contention that constitutive relations such as those for Navier–Stokes fluids wherein an expression is provided for the stress in terms of the kinematics have the reasoning for specifying constitutive relations topsy-turvy. A central dictate of classical Newtonian mechanics is the notion of causality and thus any modeling that is carried out within the discipline needs to take cognizance of the same. Let us suppose we have identified that the quantities that are necessary to describe the response of a body are the stress which acts on the body (the cause) and the velocity gradient that is engendered (the effect)[2]. We then have to somehow prescribe a *relation* between the cause and the effect, and here by relation we mean precisely a binary operation $R(c,e)$ where c and e stand for typical elements in the set $\mathbb{C}$ of causes and set $\mathbb{E}$ of effects. If one thinks a simpler specification might suffice, say an explicit expression where one of the two is to be expressed as a function of the other, it would seem eminently sensible to seek to express the effect as a function of the cause; expressing cause in terms of effect is hyperbatic. In more modern aspects of physics such as Quantum Mechanics, and dysteleologic theories in biology, the notion of causality is forsaken; I however do not think this is so for classical Newtonian physics. In general, most models or theories in natural philosophy can only provide relations between members of an appropriate event world; one might not be able to identify a member of the event world as cause and another as the effect[3]; thus one may have to be satisfied with only a relation in the mathematical sense of the word for describing the response of a body. Appealing to the new approach of interpreting the Navier–Stokes fluid, I show that the Stokes assumption is unreasonable for all fluids, including monatomic gases wherein it is supposed to hold.

Newton [31] is unequivocal about the fact that force is the cause and motion is the effect as evidenced by the following transpicuous sentiments that he expresses in his immortal Principia:

"*The* ***causes***[4] *by which true and relative motion are distinguished, one from the other, are the forces impressed upon bodies to generate motion.*" and "*The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*"

While Newton was interested in the motion of particles, Hooke (see reprinting in Gunther, [9]) while discussing the response of springy bodies makes the following remarks in his famous book, Lectures De Potentia Ristitutiva or of Spring Explaining the Power of Springing Bodies:

"*About two years since I printed this Theory as an anagram at the end of my Book of the Descriptions of Helioscopes, viz ceiiinosssttuu, idest, Ut tension sic vis; That is, the Power of any Spring is in the same proportion of the tension thereof: that is if one Power stretch or bend it one space, two will bend it two, three will bend it three and so forward.*"

Here, Hooke is using the terminology "Power" to mean what we now refer to as "Force", and it is clearly the "cause" that results in the "bending" that is the effect.

A similar sentiment with regard to the causal nature of traction and hence the stress, kinematics being the "effect", espoused by Truesdell [61] is shared by many serious scholars in the field (see Rajagopal [39,41] for a detailed discussion of the role of causality in constitutive specifications in continuum mechanics):

"*A constitutive equation is a relation between forces and motions. In popular terms, force is applied to a body to "cause" it to undergo a motion, and the motion "caused" differs according to the nature of the body. In continuum mechanics the forces of interest are contact forces, which are specified by the stress tensor* **T**."

However, immediately after espousing such a view point, they provide a constitutive expression for the stress in terms of kinematics. Let us assume that to describe a fluid such as water undergoing a certain class of flows (say what is usually understood as laminar flows), the cause is the stress and the effect is the velocity gradient. Let us further suppose that we are interested in a reasonably simple constitutive specification and hence we wish to express one of the variables in terms of the other. Then, it would seem reasonable that we try to express the velocity gradient in terms of the stress, rather than the stress in terms of the velocity gradient. However, unfortunately this is not what is done in classical fluid mechanics. In marked contrast, in classical linearized elasticity and linearized viscoelasticity theory, one can, and one does, express the stress in terms of the linearized strain or vice-versa[5]. Were we to express the velocity gradient in terms of the stress, we would find that the theory is much more elegant with regard to the enforcement of constraints such as incompressibility and assumptions such as the Stokes assumption that has been a matter of much controversy would have not even arisen. As we shall see later, the Stokes assumption[6] implies that one cannot invert the classical Navier-Stokes representation and express the symmetric part of the gradient as a function of the stress.

[2] The genius of Newton lay in recognizing that force (cause) and acceleration (effect) are related and that force is not related to not some other kinematical quantity, say velocity; the specific form that the second law takes being secondary. It is also important to recognize that Newton's statement when appropriately generalized represents a basic law of physics, the balance of linear momentum. However, the statement can also be regarded as a constitutive relation; it describes how a particle responds on the application of a force (see Rajagopal [39] for a more detailed discussion of this issue). In continuum mechanics, the applied forces (the applied traction) on the body causes the contact forces (contact traction or reaction traction) within the body, which is related to the stress (reaction stress) within the body. The contact forces within the body causes the deformation of the body. However, since the contact forces implies the existence of stress (or put differently as the stress is always coexistent with the contact traction) we can think of the stress as the cause and the deformation as the effect. Of course, one could maintain that the stress and the deformation are both effects of the contact traction. However, the manner in which the two are a consequence of the contact traction is very different. When considering "causes" one always faces the difficulty of having to deal with what might be the immediate cause, otherwise having to contend *ad infinitum* with the question of the "ultimate cause". We shall not get into such philosophical disputations here. If the reader does not find the discussion concerning cause and effect convincing, he can just skip and move forward to the part wherein a different point of view is presented with regard to the development of constitutive relations, namely move on to Section 2.

[3] See Rajagopal [39] for a detailed discussion of the relevant issues.

[4] Emphasis on the word "causes" has been placed by me.

[5] If one expresses the linearized strain in terms of the stress in linearized elasticity, one finds that the material moduli that appear in the expression are the Young's modulus and Poisson's ratio and both these physically meaningful material moduli can be determined by carrying out a simple experiment. On the other hand, expressing the stress in terms of the linearized strain, one finds that the material moduli that appear in the representation are the Lamé constants, and one of them cannot be measured but only inferred. Of course, the bulk modulus, which is a combination of the two Lamé constants, can be measured directly.

[6] I hope to provide irrefutable evidence to the fact that it is totally unnecessary to appeal to the Stokes assumption to obtain appropriate models for linearly viscous fluids. In fact, the assumption is invalid.