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Non-linear vibration of hyperelastic axisymmetric solids by a mixed *p*-type method

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ABSTRACT

This paper presents finite amplitude transient vibration analysis of nearly incompressible hyperelastic axisymmetric solids by a mixed *p*-type method. In this method, displacement and pressure fields are separately defined using high degree polynomials and the solution is obtained with one or a few elements depending upon the nature of the problem. Geometry of the element is defined by polynomials of degrees much lower than that of displacement fields. The degrees of polynomials for pressure fields are lower than those used for displacement fields.

Hyperelastic material is modelled by the Mooney–Rivlin material description. The total Lagrangian formulation is utilized to describe the deformations of axisymmetric solids subjected to pressure loads. Equations of motion are derived using the principle of virtual work and solved by the Newmark's method along with the Newton–Raphson iterative technique. The present formulation also includes the asymmetric tangent load matrix, resulting in linearization of deformation dependent load, which greatly reduces the number of equilibrium iterations to get the convergence of results at large strains.

A convergence study of the results is presented with respect to the degrees of polynomials for displacement and pressure fields. The present method is verified by successfully comparing the results with those from finite element method using the commercial software ANSYS. The numerical simulations are conducted on circular plate, solid cylinder and spherical shells subjected to time dependent pressure loads and the highly non-linear behaviour of hyperelastic solids undergoing finite amplitude vibrations is studied. The method, presented herein, is very efficient, locking free, and accurate, which does not require a large number of elements or the total degrees of freedom as it is required in conventional finite element method for the convergence of results.

1. Introduction

Hyperelastic materials are quite common in many engineering applications. These materials are incompressible or almost incompressible and undergo large strains when subjected to loads. In the last six decades, many constitutive models [1–3] are developed for hyperelastic materials, which can be used in computational model according to the application. Nevertheless, computational modelling poses challenges owing to incompressibility. For example, the displacement based finite element methods, which are widely used for various applications and materials, are not efficient for almost incompressible materials [4]. The common problems with these methods, when Poisson's ratio approaches 0.5, are the incorrect stresses, the ill conditioning of stiffness matrix, and the locking. A huge amount of work has been done by many researchers in the past to deal with the problems due to the condition of incompressibility in solids. A comprehensive survey on the finite element methods of incompressible or almost incompressible materials can be found in papers by Gadala [5], Sussman and Bathe [6] and Mackerle [7,8]. In the present work, only the most relevant papers are briefly discussed.

A great deal of research work has been devoted to the mixed variational finite element methods for the analysis of incompressible and almost incompressible materials. In the mixed method, the displacements and the stresses (or hydrostatic pressures) are separately defined and are subject to variation. In the mid 1960, Herrmann [9] published first paper on the mixed variational method for incompressible and almost incompressible isotropic materials, where, in addition to displacements, mean-stresses were separately interpolated. The analysis of orthotropic materials based on the same approach was later presented by Taylor et al. [10] and Key [11]. The work by Key [11] was also applicable to non-linear analysis. Following this, many papers were published on the non-linear analysis. Oden and Key [12] wrote a paper on the non-linear analysis of axisymmetric rubber solids.

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Scharnhorst and Pian [13] and Murakawa and Atluri [14] proposed a hybrid finite element method for the analysis of incompressible solids. A detailed study on the non-linear mixed formulation for almost incompressible elastic and inelastic analysis of materials was presented by Sussman and Bathe [6]. Finite plane strain deformations of incompressible rubber-like materials were analysed by Batra [15]. Srinivasan and Peruchhio [16] studied non-linear analysis of anisotropic rubber solid using displacements, stresses, and strains as variables. Elastomeric butt-joints were studied by Kakavas et al. [17] by mixed finite element method.

The reduced/selective integration approach is another research area that was developed in the mid 1970s, which has the simplicity of the displacement based finite element method. In this approach, the volumetric terms in the stiffness matrix are integrated by low order of Gauss integration. Malkus and Hughes [18] reported that the reduced/selective integration approach, under certain conditions, is equivalent to the mixed formulation. Mixed formulation with displacements, pressure, and volume ratio as independent variables and reduced/selective integration approach for large strain analysis of almost incompressible hyperelastic solids were studied by Papoulia [19]. In his work, it was reported that the results converged at higher rate with mixed formulation than with reduced/selective integration method and also, the two methods are not equivalent in large strain analysis.

There are issues with the stability and convergence of numerical results in the mixed finite element formulations. Any arbitrary number of primary and secondary variables cannot be used [4]. Brezzi [20] and Babuška [21,22] independently conducted a mathematical investigation of the issues with the stability of the mixed finite element formulations and derived the inf–sup condition for the safe selection of the primary and secondary variables. Nevertheless, it is a very complex procedure to check the Brezzi–Babuška stability condition for the mixed finite elements [6,23]. Zienkiewicz et al. [24] presented the counting rules and the patch test as a simple guideline to select the number of primary and secondary variables. Similarly, a heuristic constraint counting scheme for selecting the primary and secondary variables was reported by Hughes [23].

Düster et al. [25] presented a displacement based p-version of finite element method for large strain isotropic hyperelastic materials. They reported that the high-order elements are not highly sensitive to distortion. The analysis of hyperelastic shell structures is presented by Basar et al. [26] using a general displacement based high-order p-finite element formulation for finite strains and rotations. They used high-order hierarchical shell models and noted that high order formulation shows less locking.

All the work discussed above is concerned with the static analysis of incompressible or almost incompressible solids. A very little relevant amount of work is available in the literature on the vibration analysis of hyperelastic materials. Wielgosz and Marckmann [27] studied the non-linear dynamic analysis of almost incompressible hyperelastic materials under impact loadings by the finite element method based on the mixed variational principle. Dynamics of very soft tissues were analysed by Miller et al. [28]. In their paper, an algorithm based on the finite element method using total Lagrangian formulation and explicit time integration was proposed.

The dynamic analysis of hyperelastic materials is restricted to only h-type finite element methods including the mixed method. The mixed p-type method, to the best of authors' knowledge, has never been examined for the large strain dynamic analysis of almost incompressible hyperelastic solids.

In the present study, a mixed p-type method is proposed for finite strain transient vibration analysis of almost incompressible hyperelastic axisymmetric solids. The hydrostatic pressure fields, in addition to the displacement fields, are separately interpolated in terms of shape functions using polynomials of high degrees. This method requires only a few elements to get the convergence of the results. The geometry of the element is carefully defined by polynomials whose degrees depend on the shape of the geometry [29]. The degrees of polynomials to define the displacement fields are much higher than that of geometry of the element. For the development of mixed p-type elements, continuous pressure field with pressure nodes defined inside and on the element boundaries are used. The pressure fields are generally defined by lower degree polynomials than those used for the displacement fields.

The Mooney–Rivlin material description is utilized for almost incompressible hyperelastic material. The total Lagrangian formulation is adopted for describing deformations of axisymmetric solids of revolution. Equations of motion are derived by the principle of virtual work and solved by the Newmark's method along with the Newton–Raphson iterative technique. The axisymmetric solids are subject to pressure loads, which are deformation dependent. The asymmetric tangent load matrix that results in linearization of the deformation dependent loads is also included in the formulation, which greatly reduces the number of equilibrium iterations required to get the convergence of results at large strain. A convergence study of the numerical results with respect to the degrees of polynomials for displacement and pressure fields is presented. In order to verify the present method, results are successfully compared with those from h-type mixed finite element method using the commercial software ANSYS [30]. Clamped circular plate, solid cylinder, and spherical shells subjected to time dependent pressure loads are analysed. Results are presented in terms of displacement time histories and deformed shapes and the highly non-linear behaviour of hyperelastic solids is examined. The mixed p-type method is very efficient as it gives accurate results without locking at high convergence rate. Unlike conventional mixed h-type finite element method, the mixed p-type method requires only a few elements depending upon the nature of the problem to obtain accurate results.

2. Total Lagrangian formulation for axisymmetric solids

The total Lagrangian formulation is adopted for describing the deformation of axially symmetric solids of revolution in terms of cylindrical coordinate system (R,Z,θ). The solid of revolution is generated by revolving a quadrilateral region, shown in Fig. 1 in

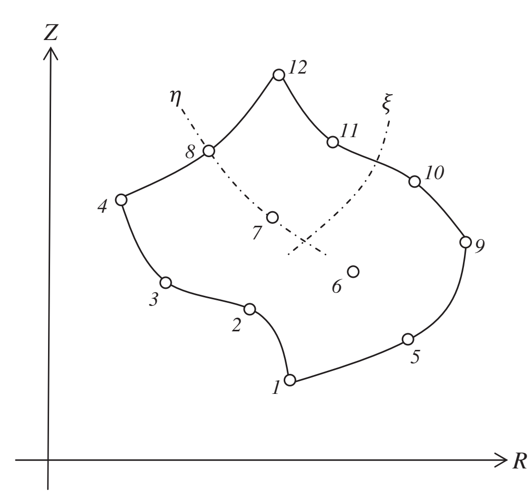

Fig. 1. Arbitrary shaped quadrilateral region.