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Non-rotating beams isospectral to a given rotating uniform beam

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ABSTRACT

In this paper, we seek to find non-rotating beams with continuous mass and flexural stiffness distributions, that are isospectral to a given uniform rotating beam. The Barcilon–Gottlieb transformation is used to convert the fourth order governing equation of a non-rotating beam, to a canonical fourth order eigenvalue problem. If the coefficients in this canonical equation match with the coefficients of the uniform rotating beam equation, then the non-rotating beam is isospectral to the given rotating beam. The conditions on matching the coefficients leads to a pair of coupled differential equations. We solve these coupled differential equations for a particular case, and thereby obtain a class of non-rotating beams that are isospectral to a uniform rotating beam. However, to obtain isospectral beams, the transformation must leave the boundary conditions invariant. We show that the clamped end boundary condition is always invariant, and for the free end boundary condition to be invariant, we impose certain conditions on the beam characteristics. We also verify numerically that the frequencies of the non-rotating beam obtained using the finite element method (FEM) are the exact frequencies of the uniform rotating beam. Finally, the example of beams having a rectangular cross-section is presented to show the application of our analysis. Since experimental determination of rotating beam frequencies is a difficult task, experiments can be easily conducted on these rectangular non-rotating beams, to calculate the frequencies of the rotating beam.

1. Introduction

An undamped vibrating system has a set of natural frequencies, forming a spectrum, at which it vibrates freely without external forces. An important class of problems for such vibrating systems can be broadly classified into inverse problems and isospectral problems. In inverse problems [2,3], one tries to determine the material properties of a system for a given frequency spectrum. In general, more than one frequency spectrum is required for a reconstruction procedure to determine the material properties of the system [4]. Systems, that have the same vibrating frequencies, but have different material properties are called isospectral systems. Isospectral systems are of great interest in mechanics as they yield alternative usable designs. The existence of isospectral systems also proves that a system cannot be uniquely identified from its spectrum.

1.1. Isospectral spring–mass systems

Examples of discrete isospectral systems are in-line spring–mass systems. A schematic of a 2-DOF spring mass system is shown in Fig. 1 , where m_1,m_2 are the masses and k_1,k_2 are the spring constants of the springs. Let us consider one such 2-DOF system (System-*A*) [1] where the values of masses m_1 and m_2, and spring constants k_1 and k_2, and the natural frequencies of System-*A* (ω_1 and ω_2) are tabulated in Table 1. Similarly, let us consider one more system (System-A') where m'_1 and m'_2 are the masses, k'_1 and k'_2, are the spring constants and ω'_1 and ω'_2 are the natural frequencies, whose values are tabulated in Table 1. From Table 1, we can see that both System-*A* and System-A' have the same frequency spectrum as $\omega_1 = \omega'_1$ and $\omega_2 = \omega'_2$. Therefore, System-*A* and System-A' are isospectral.

Gladwell [5] described four ways to form in-line spring mass systems isospectral to a given one. Gladwell [6] also considered a discrete model of a vibrating cantilever beam, and presented two procedures for finding families of such beams, isospectral to a given one. Gottlieb [7,8] studied isospectral vibrating strings with discontinuous coefficients. All these studies addressed discrete models of vibrating beams.

Borg [9] studied isospectral systems corresponding to a vibrating string with continuous coefficients (second order governing equation). Gottlieb [10] analyzed the non-uniform Euler–Bernoulli beam equation and gave seven classes of non-uniform beams isospectral to the given uniform beam with different boundary conditions. Subramanian and Raman [11] generalized Gottlieb's method for tapered beams. However, studies on isospectral systems have not

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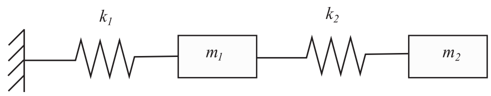

Fig. 1. Schematic of a 2DOF spring–mass system.

Table 1
Material and spectral properties of System-*A* and System-*A*′.

System-*A*		System-*A*′	
m_1	6.25 kg	m'_1	1.286 kg
m_2	4 kg	m'_2	1 kg
k_1	92.5 kN/m	k'_1	15.857 kN/m
k_2	20 kN/m	k'_2	6 kN/m
ω_1	62.2 rad/s	ω'_1	62.2 rad/s
ω_2	138.3 rad/s	ω'_2	138.3 rad/s

addressed rotating beams. In this study, we seek to find non-uniform beams that are isospectral to a given rotating beam. Such beams, if they exist, have important applications in the dynamics of rotating systems.

Rotating beams serve as a useful mathematical model to simulate vibration of helicopter blades, wind turbines, long flexible rotating space booms, turbo-machinery blades, etc. A rotating mechanical system can suffer from high vibration, if its natural frequencies coincide with multiples of the rotation speed. Therefore, an accurate determination of the frequencies is an important aspect in rotating blade design. The natural frequencies and mode shapes can be determined using an approximation scheme such as the Rayleigh–Ritz method [12], Galerkin method [13], finite element method [14–19], differential transform method [20,21] or the dynamic stiffness method [22]. Hodges and Rutkowski [23] analyzed the out of plane vibrations of a rotating beam using a finite element method of variable order. Wright et al. [24] presented an accurate solution for the mode shapes of a beam attached to a rotating hub using the method of Frobenius. Storti and Aboelnaga [25] listed the classes of beams which admit hypergeometric solutions to the mode shape equation. Naguleswaran [26] solved the mode shape equation using the method of Frobenius. Low [27] developed an algorithm for solving frequency equations for a cantilever double-span non-rotating beam and a uniform rotating beam. Thus, we see that considerable amount of research has been done on the models of a rotating beam.

Experimental determination of rotating beam frequencies can be difficult. For example, Senatore [28] experimentally determined the frequencies of a rotating beam, using lumped parameter axial loads on a uniform non-rotating beam, in order to simulate an approximation of the centrifugal force field, acting on the rotating beam. However, this method could not predict exactly the first natural frequency and mode shape due to the lumped axial loads. Hence, it is interesting to see, if we can find any non-rotating beam which is isospectral to the uniform rotating beam. It is important to take into account the exact centrifugal force acting on the rotating beam. The reasons for obtaining isospectral beams are the following. (a) It is difficult to conduct experiments on rotating beams to obtain its natural frequencies. If we find an isospectral non-rotating beam, one can easily conduct experiments on the non-uniform beam, to obtain the natural frequencies. (b) Such isospectral beams can provide insight by predicting the stiffening effect of the centrifugal force. (c) Provide benchmark problems for finite element analysis.

In this paper, we find non-rotating beams with continuous mass and flexural stiffness distributions, that are isospectral to a given uniform rotating beam. The mass and stiffness functions of non-rotating beams, isospectral to a uniform beam rotating at different speeds are derived. We note that for high rotating speeds, the derived mass and stiffness functions of the non-rotating beam are not physically realizable, owing to stiffening effect of the centrifugal force. In such situations, if we attach a torsional spring, of a spring constant K_R, at the free end of the non-rotating beam, the obtained mass and stiffness functions become physically realizable. We also show numerically that the frequencies of the non-rotating beam obtained using the finite element method (FEM) are the exact frequencies of the uniform rotating beam. This confirms numerically that the non-uniform beam obtained in this method is isospectral to the given uniform rotating beam. We also provide an example of realistic beams having a rectangular cross-section to show a physically realizable application of our analysis.

2. Mathematical analysis

In this section, the mathematical formulation of the problem is presented. The governing differential equation for the transverse free vibrations $V(Z)$ of a uniform rotating beam (Fig. 2) of length L, stiffness EI_0 and mass M_0 rotating with an angular speed Ω is given in [23] as:

$$EI_0\left[\frac{d^4V}{dZ^4}\right]-\frac{d}{dZ}\left[M_0\Omega^2\left(\frac{L^2-Z^2}{2}\right)\frac{dV}{dZ}\right]-\omega^2M_0V=0,\quad 0\le Z\le L \tag{1}$$

We introduce a non-dimensional variable $z=Z/L$ so that the above equation can be rewritten as

$$\frac{d^4V}{dz^4}-\frac{d}{dz}\left[\lambda^2\left(\frac{1-z^2}{2}\right)\frac{dV}{dz}\right]-\eta^2V=0,\quad 0\le z\le 1 \tag{2}$$

where η is the non-dimensional frequency given by

$$\eta=\omega\sqrt{M_0L^4/EI_0} \tag{3}$$

and λ is the non-dimensional rotation speed given by

$$\lambda=\Omega\sqrt{M_0L^4/EI_0} \tag{4}$$

Similarly, the governing equation for the out of plane free vibrations $Y(X)$ of a non-uniform non-rotating Euler–Bernoulli beam (Fig. 3), which is isospectral to the rotating beam is given by [29] as

$$\frac{d^2}{dX^2}\left[EI(X)\frac{d^2Y}{dX^2}\right]-\omega^2M(X)Y=0,\quad 0\le X\le \overline{L} \tag{5}$$

where $EI(X)$ is the flexural stiffness , $M(X)$ is the mass/length, $\overline{L}$ is the length of the beam, which is same as that of the uniform rotating beam ($\overline{L}=L$). Now, we introduce non-dimensional variables f, m and x as

$$f(x)=\frac{EI(X)}{EI_0},\quad m(x)=\frac{M(X)}{M_0},\quad x=\frac{X}{L} \tag{6}$$

Eq. (5) can be rewritten as

$$(f(x)Y'')''=\eta^2m(x)Y,\quad 0\le x\le 1 \tag{7}$$

Here, the notation Y' and Y'' represent the first and the second derivatives respectively of Y w.r.t x. The transformation, which

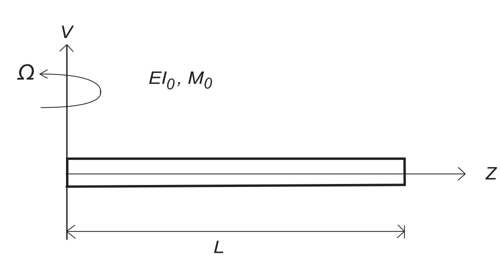

Fig. 2. Schematic of a uniform rotating beam.