International Journal of Non-Linear Mechanics 47 (2012) 1–6

Contents lists available at SciVerse ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Analytical method for the construction of solutions to the Föppl–von Kármán equations governing deflections of a thin flat plate

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ARTICLE INFO

Article history:
Received 19 October 2010
Received in revised form
16 January 2012
Accepted 18 January 2012
Available online 28 January 2012

Keywords:
Föppl–von Kármán equations
Deflections of thin flat plates
Non-linear PDEs
Perturbation methods
Homotopy analysis
Discrete residual error analysis

ABSTRACT

We discuss the method of linearization and construction of perturbation solutions for the Föppl–von Kármán equations, a set of non-linear partial differential equations describing the large deflections of thin flat plates. In particular, we present a linearization method for the Föppl–von Kármán equations which preserves much of the structure of the original equations, which in turn enables us to construct qualitatively meaningful perturbation solutions in relatively few terms. Interestingly, the perturbation solutions do not rely on any small parameters, as an auxiliary parameter is introduced and later taken to unity. The obtained solutions are given recursively, and a method of error analysis is provided to ensure convergence of the solutions. Hence, with appropriate general boundary data, we show that one may construct solutions to a desired accuracy over the finite bounded domain. We show that our solutions agree with the exact solutions in the limit as the thickness of the plate is made arbitrarily small.

1. Introduction

The Föppl–von Kármán equations, a set of non-linear partial differential equations describing the large deflections of thin flat plates, read

$$D\nabla^4 w - h\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right) = P, \tag{1a}$$

$$\nabla^4 F + E\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\} = 0, \tag{1b}$$

where $w=w(x,y)$ is the out of plane deflection, $F=F(x,y)$ is the Airy stress function, E is Young's modulus, h is the thickness of the plate,

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

is called the flexural (or, cylindrical rigidity, or, bending stiffness, in various literature) of the plate, ν is Poisson's ratio, and P is the external normal force per unit area of the plate. Furthermore, ∇^4 denotes the biharmonic operator

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}.$$

There is a long history to these equations (see, e.g., [1–4]). While the numerical and experimental study of these equations has been well represented in the literature, few analytical results have been reported. Chen and Hutchenson [5] and Huang et al. [6] recently conducted analysis on the equations, under certain specific assumptions and special cases. Regarding the buckling of the plate, Audoly [7] performs a weakly-non-linear analysis above the buckling threshold, and the results are compared to numerical simulations. Analytical solutions in the case of the straight-sided blister governing by the Föppl–von Kármán equations is given in [8–10]. Mathematically, such results correspond to an infinite strip with a sinusoidal profile. Further results in the form of sinusoidal functions are given for the herringbone pattern in [11,12]. From a purely mathematical point, the Föppl–von Kármán equations have been studied by Knightly in [13], who established a priori estimates and a global existence theorem, and furthermore showed that for small data the solution is unique.

Due to the fact that the Föppl–von Kármán equations involve high order derivatives, along with two types of deformations [14], numerical solutions are typically obtained, as the problem is often too challenging for analytical methods. However, the Föppl–von Kármán equations describe the large deformations of a plate, and hence are of interest to those studying deformations and wrinkling of surfaces [3,15–18] or even blistering of such surfaces [19]. For these reasons, a method of obtaining approximate analytical solutions would be of interest.

Let us normalize appropriately so that we may consider the Föppl–von Kármán equations over the square domain

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doi:10.1016/j.ijnonlinmec.2012.01.004

$\Omega=[0,1]\times[0,1]$. In order to solve the Föppl–von Kármán equations, we need to impose boundary conditions, and these are determined by the particular application at hand. Here, we shall take the boundary conditions describing a clamped edge, although the method can be applied to the loosely clamped edge, elevated boundary, and oscillatory boundary conditions, amongst others. The clamped edge boundary conditions read

$$w=\frac{\partial^2 w}{\partial x^2}=0 \text{ and } F=\frac{\partial^2 F}{\partial x^2}=0 \text{ at } x=0,1, \tag{2a}$$

$$w=\frac{\partial^2 w}{\partial y^2}=0 \text{ and } F=\frac{\partial^2 F}{\partial y^2}=0 \text{ at } y=0,1, \tag{2b}$$

In the present brief paper, we apply the method of homotopy analysis in order to construct approximate analytical solutions to the Föppl–von Kármán Eq. (1) subject to clamped edge boundary conditions (2). In particular, the method allows us to construct perturbation solutions around the homotopy embedding parameter, q, which serves as the perturbation parameter. Then, the resulting linear equations are solved over the square domain via Fourier analysis. Importantly, we discuss a manner of error control involving two convergence control parameters, which permits us to construct approximate solutions of low order with minimal error. This is important, as the computation of higher order terms is computationally intense (due to the fact that the linear PDEs governing the higher order deformation equations are biharmonic and inhomogeneous).

2. Linearization and construction of perturbation solutions

In order to construct analytical approximations to the Föppl–von Kármán Eq. (1) subject to the clamped edge boundary conditions (2), we will proceed along the lines of the method of homotopy analysis. See the Refs. [20–27]for details; for brevity we omit certain detailed discussions which can be found in those references, and the references therein. The method of homotopy analysis has recently been applied to the study of a number of non-trivial and traditionally hard to solve non-linear differential equations, for instance non-linear equations arising in heat transfer [28–31], fluid mechanics [22,32–38], solitons and integrable models [39–42], nanofluids [43,44] and the Lane-Emden equation which appears in stellar astrophysics [45–48], to name a few areas. We shall now apply the homotopy analysis method in order to obtain approximate solutions to the Föppl–von Kármán equations. First we outline the method. Then, we demonstrate how one can obtain the terms in the approximate solutions for the clamped edge boundary data iteratively. As one obtains more terms in the approximation, one expects better accuracy in the solutions. To this end, we discuss exactly how one can study the error in the approximations using residual errors. We can control the residual errors by means of convergence control parameters, which adjust the manner of convergence of the obtained series solutions. Selecting these parameters appropriately, we can minimize the residual error of a finite term approximation to the Föppl–von Kármán equations.

First, let us define the auxiliary linear operators

$$L_1[w]=D\nabla^4 w, L_2[F]=\nabla^4 F, \tag{3}$$

and then construct the homotopies

$$\begin{aligned}H_1(\hat{w}(x,y,q),\hat{F}(x,y,q);q)&=(1-q)L_1[\hat{w}(x,y,q)-w_0(x,y)]\\&\quad-qc_1N_1[\hat{w}(x,y,q),\hat{F}(x,y,q)],\\H_2(\hat{w}(x,y,q),\hat{F}(x,y,q);q)&=(1-q)L_2[\hat{F}(x,y,q)-F_0(x,y)]\\&\quad-qc_2N_2[\hat{w}(x,y,q),\hat{F}(x,y,q)]\end{aligned} \tag{4}$$

where N_1 and N_2 denote the original non-linear operators

$$N_1\left[\hat{w},\hat{F}\right]=D\nabla^4\hat{w}-h\left(\frac{\partial^2\hat{F}}{\partial y^2}\frac{\partial^2\hat{w}}{\partial x^2}+\frac{\partial^2\hat{F}}{\partial x^2}\frac{\partial^2\hat{w}}{\partial y^2}-2\frac{\partial^2\hat{F}}{\partial x\partial y}\frac{\partial^2\hat{w}}{\partial x\partial y}\right), \tag{5a}$$

$$N_2\left[\hat{w},\hat{F}\right]=\nabla^4\hat{F}+E\left\{\frac{\partial^2\hat{w}}{\partial x^2}\frac{\partial^2\hat{w}}{\partial y^2}-\left(\frac{\partial^2\hat{w}}{\partial x\partial y}\right)^2\right\}, \tag{5b}$$

q is the embedding parameter, both c_2 and c_2 are the convergence control parameters (which, in general, shall take different values), and both $\hat{w}(x,y,q)$ and $\hat{F}(x,y,q)$ are the solution functions which are governed by q. Furthermore, $w_0(x,y)$ and $F_0(x,y)$ are initial approximations to the solutions, which we take to satisfy the linear equations resulting from (3). When $q=0$, we have the linearization, while, for $q=1$, we have the original non-linear equations. Hence, we assume solutions to (1) and (2) of the form

$$\hat{w}(x,y,q)=w_0(x,y)+w_1(x,y)q+w_2(x,y)q^2+\cdots, \tag{6}$$

$$\hat{F}(x,y,q)=F_0(x,y)+F_1(x,y)q+F_2(x,y)q^2+\cdots, \tag{7}$$

thereby treating q as our "small parameter". Substituting (6) and (7) into the homotopies given in (4), and equating powers of q, we obtain the higher order deformation equations. The zeroth order equations read

$$L_1[w_0]=P,\quad L_2[F_0]=0, \tag{8}$$

and these are solved subject to the selected boundary conditions and inhomogeneities. The resulting functions are the initial approximations. The higher order deformation equations read

$$\begin{aligned}L_1&\left[w_\ell-\chi_\ell w_{\ell-1}\right]\\&=c_1\left\{D\nabla^4 w_{\ell-1}-h\sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1}\left(\frac{\partial^2 F_j}{\partial y^2}\frac{\partial^2 w_i}{\partial x^2}+\frac{\partial^2 F_j}{\partial x^2}\frac{\partial^2 w_i}{\partial y^2}-2\frac{\partial^2 F_j}{\partial x\partial y}\frac{\partial^2 w_i}{\partial x\partial y}\right)\right\},\end{aligned} \tag{9a}$$

$$L_2\left[F_\ell-\chi_\ell F_{\ell-1}\right]=c_2\left\{\nabla^4 F_{\ell-1}+E\sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1}\left(\frac{\partial^2 w_j}{\partial x^2}\frac{\partial^2 w_i}{\partial y^2}-\frac{\partial^2 w_j}{\partial x\partial y}\frac{\partial^2 w_i}{\partial x\partial y}\right)\right\}, \tag{9b}$$

and these are solved subject to homogeneous boundary conditions (so that the only boundary conditions come from the zeroth order terms). Note that $\chi_\ell=0$ when $\ell=0,1$ or $\chi_\ell=1$ if $\ell\geq 2$. Note also that the pressure P enters only into the order zero term for w_0, given in (8). For all higher order terms, governed by (9), only the non-linearities come into play. This is one benefit of the homotopy analysis method: inhomogeneities can be captured in the order zero terms.

We remark that, while somewhat complicated, the operators L_1 and L_2 are more representative than other simpler operators we might have taken. The benefit to selecting such operators lies in the fact that such operators are representative of the original non-linear PDEs and, hence, should permit more rapid convergence of the perturbation solutions. Note that we have not yet needed to specify boundary conditions: Indeed, the influence of boundary conditions is on (8), and the recursive relations (9) are the same for various boundary data. Hence, the method can be employed for more general boundary conditions than those considered here.

3. Recursive solutions for the clamped edge boundary data

Once the homotopies and linearizations are formulated, recovering the approximate solutions is more or less mechanical. First, let's recall that the order zero approximations satisfy (8) subject to the relevant boundary conditions. Thus, for given boundary conditions, $w_0(x,y)$ is always uniquely determined, while the natural boundary conditions are always taken in a manner such