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A transfer matrix method for free vibration analysis and crack identification of stepped beams with multiple edge cracks and different boundary conditions

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ABSTRACT

This paper illustrates an analytical approach to investigating natural frequencies and mode shapes of a stepped beam with an arbitrary number of transverse cracks and general form of boundary conditions. A new method to solve the inverse problem of determining the location and depth of multiple cracks is also presented. Based on the Euler–Bernoulli beam theory, the stepped cracked beam is modeled as an assembly of uniform sub-segments connected by massless rotational springs representing local flexibility induced by the non-propagating edge cracks. A simple transfer matrix method is utilized to obtain the general form of characteristic equation for the cracked beam, which is a function of frequency, the locations and sizes of the cracks, boundary conditions, geometrical and physical parameters of the beam. The proposed method is then used to form a system of $2N$ equations in order to identify N cracks exploiting $2N$ measured natural frequencies of the damaged beam. Various numerical examples for both direct and inverse problem are provided to validate the present approach. The results are in good agreement with those obtained by finite element and experimental methods.

1. Introduction

Dynamic behavior of machine components represents one of the main problems in solid mechanics and must be controlled to ensure the safety and reliability of structures against collapse or to assess their residual load carrying capacity. It has a crucial importance, especially for aerospace, mechanical, civil and ocean engineering. Mechanical members like beams and columns, which are widely used in high speed machinery or aircraft structures, may contain imperfections such as cracks. The cracks may develop from flaws due to applied cyclic loads, mechanical vibrations, aerodynamic loads, etc. and it is obvious that they cause a lower structural integrity and change dynamic properties such as natural frequencies and mode shapes of the components, so should be certainly taken into account in the vibration analysis of the structures.

Investigating the dynamic behavior of cracked beams has received a great deal of attention over the recent years. Dimarogonas [1] and Gasch [2] presented comprehensive reviews of various methods in tackling a cracked structural member. Dimarogonas and Paipetis [3] suggested an attractive method for modeling the open edge crack in a beam as a local flexibility which can be derived from the stress intensity factors in the theory of fracture mechanics. Under the most general loading, the local flexibility can be represented by a matrix of dimension 6×6 [4]. Local flexibility coefficients depend on the size of the crack and crack plane's geometry. In the case of transverse vibration of beams under pure bending, the cracked section may be replaced by a single rotational spring representing local flexibility of the crack [5]. Using the local flexibility for investigating free vibration of edge-cracked beams includes two aspects; the first one is the effects of the cracks on the structural dynamic characteristics like natural frequencies and mode shape of damaged beams as a "direct problem" and the second one is how to predict the location and size of the cracks from the measured information of the damaged beam system as an "inverse problem". The direct analysis of vibrating beams in the presence of edge cracks is necessary to solve the inverse problem.

Euler–Bernoulli theory has been used in many previous studies on the cracked beams with various boundary conditions [6–9]. Narkis and Elmalah [10] analyzed vibration of the simply supported cracked beam by using a massless rotational spring and dividing the beam into two sub-beams. Zheng and Fan studied beams with hollow rectangular and circular sections [11]. Lele and Maiti [12] provided a new method based on the Timoshenko theory for crack identification in beams and Loya et al. [13] studied the effect of cracks on the natural frequencies of a simply supported Timoshenko beam. In general, there are two main

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categories to study dynamic behavior of a cracked beam; continuous method and discrete method. In the continuous method, a beam is divided into a number of sub-beams (sub-segments) connected by rotational springs and differential equations are solved for each beam individually considering boundary conditions [5,14] and discrete technique involves the finite element method [15]. In the field of inverse problems, greater attention in recent years has been devoted to detecting, locating and quantifying the extent of the crack, based on changes of fundamental frequencies, mode shapes or dynamic flexibility [12,16–21]. Liang and co-workers [16] utilized frequency contour plot method exploiting the first three natural frequencies to detect a crack in a beam. Lee [17] used a combination of the finite element method and the Newton–Raphson procedure to identify multiple cracks.

Studies on the bending vibration of edge-cracked beams are mostly presented for uniform beams with only one or at most two cracks and there are a few studies on the vibration of non-uniform beams with cracks. Jang and Bert [22] analyzed free vibration of stepped beams with different boundary conditions and Naguleswaran [23] studied the vibration of a stepped beam with axial force. Nandwana and Maiti [7] presented a method for crack detection in a stepped cantilever beam. Shifrin and Ruotolo [14] proposed a technique which can be used to analyze the vibratory characteristics of a beam with multiple cracks. However, only uniform beams can be solved using their approach.

The transfer matrix method (TMM) is a prevalent and efficient tool for free vibration analysis of beams with non-uniform mechanical properties. This method, first introduced by Pestel and Leckie [24], has been the subject of several research papers [25–27]. Modified transfer matrix methods are also developed to study the dynamic behavior of beams with various attachments [28,29].

The objective of the present paper is to present an analytical method to investigate free vibration of a stepped beam having an arbitrary number of transverse open cracks with general form of boundary conditions and provide an efficient approach to solving the inverse problem of detecting multiple open cracks in a beam. Both ends of the beam carry lumped masses (m_1 and m_2) and are supported by linear (K_1 and K_2) and rotational (K_{R1} and K_{R2}) springs. Using the elastic end conditions for the beam, one may easily model all kinds of two end supports by choosing appropriate values for stiffness of the springs. The present model is based on the continuous method and the stepped cracked beam is modeled as an assembly of uniform sub-segments connected by massless rotational springs representing local flexibility induced by open non-propagating edge cracks. The flexibilities of these springs are calculated using fracture mechanics theory [11].

Based on the Euler–Bernoulli beam theory, differential equations for free vibrations are derived for each segment. Four unknown coefficients appear in the solution of deflection function for each sub-segment of cracked beam. To determine these constants the transfer matrix method is employed to satisfy the conditions at all boundary points of the sub-segments, which leads to a general frequency equation for the damaged beam. This equation is expressed in terms of the elements of the overall transfer matrix. In addition, the mode shapes of the damaged beam play a crucial role in providing the local and whole information of the structure. In addition to natural frequencies, equivalent mode shapes can be derived using the present TMM.

To solve the inverse problem, this study presents an effective scheme based on the transfer matrix method and the Newton–Raphson iteration procedure to identify N cracks exploiting $2N$ measured natural frequencies of the damaged beam. In this technique, the proposed transfer matrix method is utilized to form a system of $2N$ equations, where the locations and sizes of the cracks are unknown parameters. This system of equations may be solved by an appropriate numerical method to yield the crack parameters.

Various detailed numerical examples are also given to demonstrate the effectiveness of the proposed procedure. The results are in good agreement with those obtained by finite element and experimental methods. The calculated frequency equation and corresponding mode shapes can be useful to evaluate the influence of parameters like boundary conditions, crack depth, crack location, number of cracks and structural constants, reducing significantly computational time in comparison with a detailed finite element analysis. Moreover, it can provide a wide range of data sets which are essential for most of the crack detection procedures.

2. Local flexibility due to a crack

An open crack on an elastic structure can be considered as a source of local flexibility due to the strain energy concentration at the surrounding area of the crack tip. The idea of replacing a crack by a massless spring is presented to establish the relation between the strain energy concentration and applied loads. The flexibility coefficients are expressed in terms of stress intensity factors, utilizing Castigliano's theorem. Generalized loading conditions for a beam element of circular or rectangular cross-section with a transverse surface crack are shown in Fig. 1. The crack has a tip line parallel to z-axis and the bar is loaded with axial load, P_1, shear loads, P_2 and P_3, bending moments, P_4 and P_5, and torsional torque, P_6. According to the Castigliano's theorem, the additional displacement caused by the crack is given as

$$u_i = \frac{\partial U}{\partial P_i} = \frac{\partial}{\partial P_i}\left[\int_{A_c} J\mathrm{d}A\right] \tag{1}$$

where U is strain energy due to the crack, A_c is crack section, u_i is the additional displacement in the direction of the loading P_i and J is strain energy density function given by Tada et al. [30] as

$$J = \left(\frac{1}{E'}\right)\left[\left(\sum_{i=1}^{6} K_{Ii}\right)^2 + \left(\sum_{i=1}^{6} K_{IIi}\right)^2 + (1+\upsilon)\left(\sum_{i=1}^{6} K_{IIIi}\right)^2\right] \tag{2}$$

where v is Poisson's ratio, E is Young's modulus, $E' = E$ for plane stress and $E' = E/(1-v^2)$ for plane strain and $K_{ni}(n = I,II,III)$ is the crack stress intensity factor of mode n corresponding to the generalized loading P_i. The SIF in the Eq. (2) is determined as

$$K_{ni} = \sigma_i\sqrt{\pi a}F_{ni} \quad (n = I,II,III), \quad (i = 1,2,\ldots,6) \tag{3}$$

In which, σ_i is the stress at the crack cross-section due to ith independent force, a is the crack depth and F_{ni} denotes a geometry dependant non-dimensional crack configuration factor. Now, the flexibility coefficient, by definition, is

$$C_{ij} = \frac{\partial u_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j}\left[\int_{A_c} J\mathrm{d}A\right] \tag{4}$$

According to Eqs. (2)–(4), the elements of the local flexibility matrix depend only on the degrees of freedom being considered for the moments and forces applied on the crack section. The full

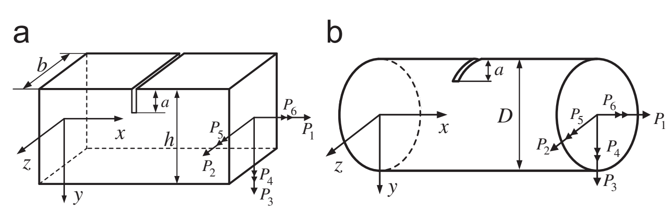

Fig. 1. Beam with an open edge crack under generalized loading condition: (a) rectangular cross-section and (b) circular cross-section.