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Mathematical and numerical modeling of the non-associated plasticity of soils—Part 1: The boundary value problem

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A B S T R A C T

The soil is characterized by the influence of the hydrostatic stress, which leads to a yield surface with a shape of a pyramid for Mohr–Coulomb criteria and a shape of a cone for Drucker–Prager one. These materials are also characterized by a non-associated plasticity where the plastic yielding rule does not follow the normality rule. The usual mechanical models use two independent functions to describe this particular collapse. Unfortunately, this manner broke the model formulation. The purpose of this work is to present a consistent formulation of the non-associated plasticity of soil. The frame of the mathematical analysis is the concept of the implicit standard material. The cornerstone of this new idea is the construction of a single function called the bipotential playing in the same time the roles of the yield surface and the plastic potential. The bipotential concept is then intended to involve the constitutive law, cover the normality rule even for the non-associated soil and the proof of the solution existence. The formulation was initially performed for the case of a regular point out of the cone apex and in present, it is extended to the irregular point located at the apex. The paper presents firstly the implicit standard material method. Then, the methodology to build a full model for the boundary value problem is detailed. Particular expressions and relations are sufficiently explained and discussed. Attention is made to the evolution problem and the variational principles related to the elastic–plastic behavior.

1. Introduction

The soils are characterized by special effects such as volume change even in plastic range, presence of softening in the load–displacement curve and a significant decrease of the limit load with respect to the corresponding standard material [1]. So, compared with metallic materials, they are non-standard solids [2] and for evident reasons, rigid perfectly plastic models were extensively used in analytical problems. One cites, slip line theory [3], limit equilibrium theory [4,5] and limit analysis [6]. In limit analysis theory, several works can be noted such as in [7]. Recently, with the progress of experimentations and numerical methods, sophisticated models have been proposed such as non-associated elastic–plastic behavior [8]. In the later model, a new parameter is introduced: the angle of plastic flow or plastic dilatancy angle θ, which lies within the range $[0, \varphi]$, where φ is the internal friction angle. The case $\theta=0$ corresponds to plastically incompressible materials and the case of $\theta=\varphi$ corresponds to associated plasticity. All cases where $\theta \in [0, \varphi]$ is non-associated in the sense that the rate of plastic flow is not normal to the level-set of yield function $f(\boldsymbol{\sigma})$. In non-associated plasticity, a second function $g(\boldsymbol{\sigma})$ called the plastic potential is usually introduced to model the flow rule independent of the yield function such that the normality rule is disabled. This approach is a simple artifice but it has the drawback to lose the consistence of the formulation. So, solution existence is not established. Also, this approach leads to a non-symmetrical system of equations. Some works have tried to recover the symmetrical system by the introduction of an artificial hardening [9,10] or by the use of an energy equivalence principle [11]. Presently, a more fruitful alternative is presented. Generalizing the Fenchel's inequality to materials and systems with non-standard behaviors [12], it is possible to build a new class of mixed formulation using only one function known as the bipotential. It plays at the same time the roles of potential of dissipation and the yield function. This formalism leads then to a complete model able to proof the solution existence of the boundary value problem for non-associated flow rule. To recover the key-concept of normal dissipation, the method expresses the constitutive law under an implicit relation. The material admitting a bipotential function is called implicit standard material (ISM). This approach has known relatively intensive works since its appearance in the 1990 years. Among the applications of bipotentials one cites: Coulomb's friction law [13], non-associated Drucker–Prager criterion [14]

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Nomenclature

b	bipotential function
c	cohesion
dS	surface element
dv	elementary volume
$\mathbf{e}$	strain deviator
e (index or exponent)	relating to elasticity or to finite element
$\mathbf{e}^e$	elastic part of strain deviator
$\mathbf{e}^p$	plastic part of strain deviator
e_m	trace of strain
e_m^e	trace of elastic strain
e_m^p	trace of plastic strain
f	yield function
$\mathbf{f}$	forces of volume
g	plastic potential
i (index)	relating to an iteration
k (exponent)	relating to an iteration or to kinematically admissible field
k_d	parameter
n	whole number
$\hat{\mathbf{n}}$	unit vector in deviator space
p (index or exponent)	relating to plasticity
$\mathbf{s}$	stress deviator
s (exponent)	relating to statically admissible field
s_m	trace of stress
s_{mr}	trace of stress residue
$\mathbf{s}_r$	residue of stress deviator
t	time
$\mathbf{u}$	displacement field
u (index)	relating to displacement
$\mathbf{u}^*$	approximate displacement field
$\overline{\mathbf{u}}$	imposed displacement vector
$\mathbf{x}$	generalized variable or position vector
$\mathbf{y}$	generalized variable
B	bifunctional
C_μ	convex set
C_μ^*	convex set
$\mathbf{D}$	local tangent matrix
$\mathbf{D}^e$	elasticity matrix
E	Young modulus
G	shear modulus
$\mathbf{I}$	identity matrix
K_c	bulk modulus
S	external surface
S_T	external surface liable to traction
S_u	surface liable to liaisons conditions
T (exponent)	transpose of a matrix
$\mathbf{T}$	surface tractions
V	potential function
W	complementary potential function
$\boldsymbol{\varepsilon}$	strain tensor
ε_c	constant
ε_d	constant
$\boldsymbol{\eta}$	strain deviator
θ	dilatancy angle
κ_ε	convex set for strain
κ_σ	convex set for stress
λ	Lamé constant, Augmented lagrangian multiplicator
μ	Coulomb's shear modulus
ν	Poisson's ratio
ρ	density
$\boldsymbol{\sigma}$	stress tensor
$\boldsymbol{\sigma}_r$	stress tensor residue
φ	internal friction angle
ψ_κ	convex κ indicator function
Δ	finite increment
1 (index)	relating to step end
-1 (exponent)	inverse operator
* (exponent)	relating to some quantity
, (index)	partial or total derivative
inf	inf operator
tan	tangent function
∂	differential operator
$\partial_{\mathbf{x}}$	differential operator with respect to variable $\mathbf{x}$
$\partial_{,\mathbf{x}}$	derivative operation with respect to variable $\mathbf{x}$
$\Vert . \Vert$	Euclidian norm
©$_{\mathbf{x}}$	inf convolution product with respect to variable $\mathbf{x}$
.	scalar product
$\{.\}_+$	positive part
. (accentuation)	indicate rate
′ (accentuation)	indicate some variable
ˆ (accentuation)	indicate approached quantity
- (accentuation)	indicate an imposed quantity
I.S.M.	implicit standard material
K.A.	kinematically admissible field
S.A.	statically admissible field

and Cam–Clay models [15] in Soil Mechanics. A review of other laws expressed in terms of the bipotentials can be found in [16,17]. Recently the method was extended to frictional contact with cohesive zone [18]. Newton–Raphson method is widely used in step by step computations but its efficiency is considerably decreased when a non-associated model is considered. To avoid this pitfall, a modified version was proposed for the implicit standard material approach with non-associated soil [14]. The work remains unachieved because it requires a general formulation for any stress's state on the yield surface. This is our aim in this work together the use of the solid mechanics principles to rewrite the formulation with simpler manner and more interpretations of the mathematical expressions. Lastly, all the previous papers related to the subject are based on convex analysis and generally not accessible to engineering writing. Linking between the formulation steps is not detailed. Hence, the paper tracks the following organization. A short statement of the boundary value problem specifies some notations and exposes needs in formalism. Then, the bipotential concept is recalled. Next, a detailed non-associated flow rule for soil is described using the implicit standard material method. This target constitutes the subject of several sections having as final purpose the construction of the incremental elastoplastic bipotential function. Then, the question of the solution existence is raised, which conducts to the development of a generalized minimum principle using the new concept of the bifunctional.

2. The boundary value problem

Let us consider a solid of volume V, with external surface S and density ρ. The solid is subjected to forces of volumes $\mathbf{f}$ and surface forces $\overline{\mathbf{T}}$ acting on S_T a part of the external surface. Cauchy stress principle stipulates appearance of internal surface forces $\boldsymbol{\sigma}=(\sigma_{ij})$ ensuring the transmission of efforts and equilibrium of solid such that stress field $\boldsymbol{\sigma}$ is statically admissible. Here, for practical