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Discontinuity-induced bifurcations in systems with impacts and friction: Discontinuities in the impact law

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A B S T R A C T

This paper concerns the non-smooth dynamics of planar mechanical systems with isolated contact in the presence of Coulomb friction. Following Stronge [Impact Mechanics, Cambridge University Press, Cambridge, 2000], a set of closed-form analytic formulae is derived for a rigid-body impact law based on an energetic coefficient of restitution and a resolution of the impact phase into distinct segments of relative slip and stick. Thus, the impact behavior is consistent both with the assumption of Coulomb friction and with the dissipative nature of impacts. The analysis highlights the presence of boundaries between open regions of initial conditions and parameter values corresponding to distinct forms of the impact law and investigates the smoothness properties of the impact law across these boundaries. It is shown how discontinuities in the impact law are associated with discontinuity-induced bifurcations of periodic trajectories, including non-smooth folds and persistence scenarios. Numerical analysis of an example mechanical model is used to illustrate and validate the conclusions.

1. Introduction

There has been much interest in using non-linear dynamical systems theory to understand the complex behavior of rigid body mechanics in the presence of non-smooth effects such as dry friction and impact (e.g., [4,25,28,34,38] and references therein). One difficulty is that the so-called geometric theory of dynamical systems [17,22] typically assumes that the dynamics in question is sufficiently smooth, whereas phenomena such as chattering of impacting systems [5,30] and stick–slip vibrations in the presence of Coulomb friction [35] are fundamental consequences of non-smoothness. For non-smooth mechanical systems, even basic questions like existence and uniqueness of solutions to model equations remains an area of active research, and various different formalisms exist, such as sliding modes [14], complementarity [18,37], hybrid systems [36] and differential inclusions [27].

The idea that interaction with discontinuities in a dynamical system can cause qualitative changes in the dynamics has been known for some time, see for example the pioneering work or Feigin [13]. Recently, the present authors and their collaborators have introduced the notion of a *discontinuity-induced bifurcation* as a useful paradigm for explaining dynamical phenomena that are unique to non-smooth systems. In the context of the dynamics of systems undergoing frictionless impact, Nordmark [29] (see also [15]) introduced the notion of a *discontinuity map* that is able to analytically account for the correction to the smooth dynamics induced by a *grazing* incidence with a discontinuity surface, for which he was able to show the onset of period-adding sequences and chaotic dynamics. Later Dankowicz and Nordmark [6] (see also [8]) generalized the concept to piecewise-smooth continuous models with application to models of dry friction with additional intrinsic degrees of freedom. di Bernardo et al. [12] further derived discontinuity maps for bifurcations unique to discontinuous dynamics that can undergo the so-called sliding motion (equivalent to relative stick in the present context of dry friction). Such sliding bifurcations have been shown to underlie the onset of stick–slip oscillations in a variety of models containing dry friction; see Merillas et al. [26] for the most comprehensive results to date. These techniques have also been incorporated into numerical software for simulation and parameter continuation [20,33,41]. A comprehensive theory is therefore emerging, as has been summarized in the recent book [9] and review [10], and includes application to models that include both impact and friction, see e.g. [7,40,43].

So far, the case of impacts that involve friction has not been systematically analyzed in the context of discontinuity-induced bifurcation. Note however the work by Leine et al. [24], who studied a variant of the classical Painlevé example [31] of a falling rod, albeit

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with zero coefficient of restitution. There it was shown that passage into the region in which the classical Painlevé paradox applies is associated with bifurcations of branches of equilibria and periodic orbits. Also, Lancioni et al. [23] considered simulations of a similar model (which is also closely related to the example introduced in Section 2.3 although with a rather different form of impact law) with a non-zero coefficient of restitution. They found periodic and chaotic motion with intervals of stick and chatter-type motion.

In contrast, the present paper concerns itself in generality with bifurcations of system behavior involving phases of sustained free flight, interrupted by isolated collisional contact events, for which the associated impact laws are piecewise-smooth functions of system parameters and the system state at the onset of contact. Following the approach adopted by Stronge [38] (see also Batlle [1,2] and references therein) in the presence of dry friction, such piecewise-defined impact laws are shown to result from a decomposition of the impact phase into distinct segments of slip and stick motion. Here, the termination of the impact phase is given in terms of the *energetic coefficient of restitution* [38]. Unlike impact laws based on kinematic or kinetic coefficients of restitution, this approach is guaranteed to lead to dissipative collisions in all cases (see the discussion in Section 6 for more details).

The key point of the paper is that discontinuity-induced bifurcations can occur due to the inherent non-smoothness of the impact law across well-defined boundaries associated with changes in the sequence of stick and slip segments during the impact phase. As shown in Section 5, such changes result in at-most piecewise-smooth Poincaré mappings on neighborhoods of degenerate periodic trajectories. Specifically, mappings with a discontinuity in the first derivative are known to be associated with a catastrophic loss of stable motion and sudden jumps between different kinds of attractor, see [9,10,13] and Figs. 6 and 7.

The paper is organized as follows. Section 2 reviews the Lagrangian framework for impulsive contact at isolated points on a rigid-body mechanism and illustrates the formalism for an example system. A collection of impact mappings relating incoming and outgoing relative velocities are derived in Section 3. Boundaries between open regions of initial conditions and parameter values corresponding to distinct forms of the impact mappings are enumerated in Section 4 as are the smoothness properties of the corresponding impact law across these boundaries. Section 5 goes on to study the different kinds of discontinuity-induced bifurcations that arise from the various degrees of non-smoothness in the impact law, and to provide support for these conclusions using numerical analysis of the example mechanism. The paper ends with a discussion that puts the results into the context of previous work and provides an outlook to subsequent work.

2. Mechanical model

2.1. A Lagrangian formulation

Consider a multibody mechanism whose configuration relative to an inertial reference frame may be described in terms of a column matrix q of generalized coordinates and (possibly) the time coordinate t. Its dynamics are then governed by Lagrange's equations

$$\frac{d}{dt}\left(\partial_{\dot q}T\right)-\partial_q T=F, \tag{1}$$

where the components of the row matrices $\partial_q T$ and $\partial_{\dot q}T$ are the partial derivatives of the kinetic energy T with respect to the generalized coordinates and the generalized velocities, respectively, and where F denotes a row matrix of generalized forces.

Suppose that contact occurs between a point P on the multibody mechanism and a rigid element in its environment. Throughout the duration of contact, let $F=F_c+F_a$, where F_c represents the generalized forces associated with contact interactions and F_a represents all other generalized forces acting on the mechanism. Denote by $x(q,t)$ the transformation from the generalized coordinates to the column matrix of Cartesian coordinates of the point P relative to the inertial reference frame. It follows that

$$F_c=\lambda\cdot\partial_q x \tag{2}$$

for some row matrix λ.

There exists a positive definite, symmetric matrix M, whose entries are functions of q and t, such that

$$T=\tfrac{1}{2}\dot q^T\cdot M\cdot\dot q+\cdots,$$

where the omitted terms are at most linear in the column matrix of generalized velocities $\dot q$. From (1) and (2) it follows that

$$\ddot q=M^{-1}\cdot(\partial_q x)^T\cdot\lambda^T+\cdots, \tag{3}$$

where the omitted terms are independent of λ and are a function of F_a, q, $\dot q$, and t only. Finally, denote by $v=\partial_q x\cdot\dot q+\partial_t x$ the velocity of the point P relative to the inertial frame. In terms of the symmetric matrix

$$m^{-1}=\partial_q x\cdot M^{-1}\cdot(\partial_q x)^T, \tag{4}$$

it then follows that

$$\dot v=m^{-1}\cdot\lambda^T+\cdots, \tag{5}$$

where the omitted terms are independent of λ and are a function of F_a, q, $\dot q$, and t only.

In the case of motion constrained to a plane,

$$\partial_q x=\begin{pmatrix}c_T\\ c_N\end{pmatrix},\quad v=\begin{pmatrix}v_T\\ v_N\end{pmatrix},\quad \lambda=(\lambda_T,\lambda_N),$$

where the subscripts T and N refer to components tangential and normal to the common tangent direction at P, respectively. Suppose that $\partial_q x$ has full row rank (which would not be the case for the model considered in [23,24] at points where $\phi=\pm\pi/2$). From (4) it follows that

$$m^{-1}=\begin{pmatrix}A & B\\ B & C\end{pmatrix}$$

is positive definite, i.e., that

$$A>0,\quad C>0,\quad AC-B^2>0. \tag{6}$$

The formulation (1), (2) allows for several different modes of sustained motion on open non-zero intervals of time. Let x_N be a coordinate representing the normal distance between P and the rigid element, such that sustained free motion (with $\lambda=0$) occurs whenever $x_N>0$ for such a time interval. Assume that normal contact interactions acting at P are *compressive*, i.e., that $\lambda_N\geq 0$ and that $\lambda_N=0$ when there is no contact at P. Furthermore, suppose that the simple Amonton–Coulomb friction law

$$|\lambda_T|\leq\mu\lambda_N \tag{7}$$

applies at P for some non-negative physical constant μ, representing a *coefficient of friction*. Sustained contact then occurs on intervals for which $x_N\equiv 0$ and $\lambda_N>0$. In particular, we distinguish between sustained stick where, in addition to (7), the relative velocity between P and the instantaneous point of contact on the rigid element vanishes; and sustained slip, where equality occurs in (7). This work shall not consider the dynamics of sustained contact, but shall instead treat impulsive contact, or impact, which occurs at isolated points of time separating open intervals of free flight.