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Unsteady motion of a non-linear viscoelastic fluid

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ABSTRACT

In this paper, we study the unsteady flow of a generalized second grade fluid. Specifically, we solve numerically the linear momentum equations for the flow of this viscoelastic shear-thinning (shear-thickening) fluid surrounding a solid cylindrical rod that is suddenly set into longitudinal and torsional motion. The equations are made dimensionless. The results are presented for the shear stresses at the wall, related to the drag force; these are physical quantities of interest, especially in oil-drilling applications.

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1. Introduction

There are very few exact solutions available for the flow of Newtonian fluids in complex flow geometries (cf. [1]). There are even fewer exact solutions in the case of non-Newtonian fluids (cf. [2,42]). In mechanics, especially in fluid mechanics, exact solutions to the governing equations of motion are important for many reasons. In addition to providing and enhancing the field with aesthetic beauty of closed form solutions, exact (analytical) solutions also serve as standards or measures whereby the computational/ numerical solutions (for complicated geometries and flow conditions) can be tested for accuracy and effectiveness. As few as these exact solutions are for the steady flows of the Navier–Stokes fluid (see [3]), there are even fewer exact solutions for the unsteady flows of these fluids (see [4–6]). Unsteady flows of fluids occur in nature and in many industrial applications; the unsteadiness can be due to a variety of reasons: the unsteady motion of the boundaries, application of body forces, impulsive motion or sudden acceleration of boundaries or the body, fluctuating nature of the flow as in turbulence, application of unsteady forces (or stresses) or displacement at the boundaries. The difficulty in obtaining exact solutions for the unsteady flows of many non-linear fluids is exacerbated due to the non-linearities in the constitutive relations (see [7–9]).

Developing advanced coal-based fuel production with low pollutants is an important element for a cleaner environment and a more sustainable future. Among alternative fuels for use in existing oil-fired boilers are coal slurries, particularly coal–water mixtures (CWM) or coal–oil mixtures (COM) (see [10]). In many instances these suspensions (coal slurries) exhibit non-Newtonian characteristics, such as shear–rate dependent viscosity, thixotropy, and normal stress differences. A great deal of effort has been directed towards relating rheological properties of CWM to atomization quality. Rheological properties such as high shear viscosity, yield stress, viscoelasticity and extensional viscosity have been hypothesized as the key parameters in predicting slurry atomization quality. In general, non-Newtonian fluids differ from Newtonian fluids in at least two ways: (1) they exhibit normal stress effects, such as rod-climbing and die-swell; and (2) they exhibit shear-thinning or shear-thickening which is the decrease or increase in viscosity with increasing shear rate, respectively (cf. [11]). Both these phenomena introduce non-linearities into the equations. Power-law models have been used extensively in many areas of chemical industries, polymer processing, coal slurries based fuels, etc. Some studies indicate that the viscosity of coal–water mixtures depends not only on the volume fraction of solids, and the size distribution of the coal, but also on the shear rate, since the slurry behaves as a shear-thinning fluid (see [12–14]).

Massoudi and Phuoc [15] studied the motion of a second grade fluid due to longitudinal and torsional oscillations of a cylinder numerically. In this paper, we extend that study to the unsteady flow of a generalized second grade fluid. Specifically, we solve numerically the linear momentum equations for the flow of this viscoelastic shear-thinning (shear-thickening) fluid surrounding a solid cylindrical rod that is suddenly set into longitudinal and torsional motion. The equations are made dimensionless. The results are presented for the shear stresses at the wall, related to the drag force; these are physical quantities of interest, especially in oil-drilling applications.

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In Section 2, for the sake of brevity and completeness, we present the basic equations of motion for a single continuum. In Section 3, we discuss the constitutive model for the stress tensor $\mathbf{T}$. In Section 4, we present the unsteady equations of motion under isothermal conditions, and in Section 5, the dimensionless forms of the momentum equations are solved numerically.

2. Governing equations

The governing equations of motion, where there are no thermal, chemical, or electromagnetic effects, are the conservation of mass and linear momentum. These are

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0 \tag{1a}$$

where ρ is the density of the fluid, $\partial/\partial t$ is the partial derivative with respect to time, and $\mathbf{u}$ is the velocity vector. For an isochoric motion we have

$$\mathrm{div}\,\mathbf{u} = 0 \tag{1b}$$

Conservation of linear momentum:

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathrm{div}\,\mathbf{T} + \rho \mathbf{b} \tag{2}$$

where $\mathbf{b}$ is the body force vector, $\mathbf{T}$ is the stress tensor, and $\mathrm{d}/\mathrm{d}t$ is the total time derivative, given by

$$\frac{\mathrm{d}(.)}{\mathrm{d}t} = \frac{\partial(.)}{\mathrm{d}t} + [\mathrm{grad}(.)]\mathbf{u} \tag{3}$$

3. Constitutive relation

Perhaps the simplest model which can capture the normal stress effects (which could lead to phenomena such as 'die-swell' and 'rod-climbing', which are manifestations of the stresses that develop orthogonal to planes of shear) is the second grade fluid, or the Rivlin–Ericksen fluid of grade two [16,17]. This model has been used and studied extensively [18] and is a special case of fluids of differential type [19]. For a second grade fluid the Cauchy stress tensor is given by:

$$\mathbf{T} = -p\mathbf{1} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 \tag{4}$$

where p is the indeterminate part of the stress due to the constraint of incompressibility, μ is the coefficient of viscosity, α_1 and α_2 are material moduli which are commonly referred to as the normal stress coefficients. The kinematical tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined through

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T$$

$$\mathbf{A}_2 = \frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1$$

$$\mathbf{L} = \mathrm{grad}\,\mathbf{u} \tag{5}$$

The thermodynamics and stability of fluids of second grade have been studied in detail by Dunn and Fosdick [18]. They show that if the fluid is to be thermodynamically consistent in the sense that all motions of the fluid meet the Clausius–Duhem inequality and that the specific Helmholtz free energy of the fluid be a minimum in equilibrium, then

$$\mu \geq 0$$

$$\alpha_1 \geqq 0$$

$$\alpha_1 + \alpha_2 = 0 \tag{6}$$

It is known that for many non-Newtonian fluids which are assumed to obey Eq. (4), the experimental values reported for α_1 and α_2 do not satisfy the restriction $(8)_{2,3}$. In an important paper, Fosdick and Rajagopal [20] show that irrespective of whether $\alpha_1+\alpha_2$ is positive, the fluid is unsuitable if α_1 is negative. In particular, they showed that if it is assumed that

$$\mu > 0, \quad \alpha_1 < 0, \quad \alpha_1 + \alpha_2 \neq 0 \tag{7}$$

which as many experiments have reported to be the case "for those fluids which the experimentalists *assume* to be constitutively determined by (4), at least sufficiently well as a second order approximation" (Fosdick and Rajagopal, p. 147), then certain anomalous results follow. Fosdick and Rajagopal [20] proved a theorem which indicates that if $(7)_{2,3}$ hold, then an unusual behavioral property, not to be expected for any rheological fluid occurs, namely, "that the *larger* the viscosity, keeping everything else fixed, the *faster* that initial data is amplified in motions which take place under strict isolation." For further details on this and other relevant issues[1] in fluids of differential type, we refer the reader to the review article by Dunn and Rajagopal [19].

In an effort to obtain a model that exhibits both normal stress effects and shear-thinning/thickening, Man and Sun [21], and Man [22] modified the constitutive equation for a second grade fluid by allowing the viscosity coefficient to depend upon the rate of deformation. One of the proposed models is:

$$\mathbf{T} = -p\mathbf{1} + \mu \Pi^{m/2}\mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 \tag{8}$$

where

$$\Pi = \tfrac{1}{2}\,\mathrm{tr}\,\mathbf{A}_1^2 \tag{9}$$

is the second invariant of the symmetric part of the velocity gradient, and m is a material parameter. When $m < 0$, the fluid is shear-thinning, and if $m > 0$, the fluid is shear-thickening. A subclass of models given by Eq. (8) is the generalized power-law model, which can be obtained from Eq. (8) by setting $\alpha_1 = \alpha_2 = 0$ (see [23,24,40] for further generalizations of this model]. In this paper, we will use the generalized second grade fluid model as shown in Eq. (8) (see [25] for a recent review). It also needs to be mentioned that second grade fluids (or higher order models) raise the order of differential equations by introducing higher order derivates into the equations. As a result, in general, one needs additional boundary conditions; for a discussion of this issue, see Rajagopal [26], and Rajagopal and Kaloni [27].

4. Motion of an oscillating and rotating cylinder

The motion of an oscillating cylinder and its effects on the surrounding fluid was first investigated by Stokes [28] (see also Stokes

[1] In recent years, Rajagopal and colleagues (see for example, Rajagopal and Srinivasa [37,38]) have devised a thermodynamic framework, the Multiple Natural Configuration Theory, by appealing to the maximization of the rate of entropy production to obtain a class of constitutive relations for many different types of materials. Unlike the traditional thermodynamic approach (for example, Dunn and Fosdick [18]) whereby a form for the stress is assumed (or derived) and restrictions on the material parameters are obtained by invoking the Clausius–Duhem inequality, in this new thermodynamic framework, they assume specific forms for the Helmholtz potential and the rate of dissipation—reflecting on how the energy is stored in the body and the way in which the body dissipates it. In a recent study, Rajagopal and Srinivasa [39] have modified their theory to obtain various forms for the fluid models of differential type, specially the second grade fluid; interestingly, they also arrive at the same conclusion about the restrictions imposed on α_1 and α_2, namely that $\alpha_1+\alpha_2 = 0$.