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On the linear elasticity of porous materials

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ABSTRACT

In this work, a porous material is represented as a composite made of an isotropic matrix and spheroidal inclusions with zero stiffness, describing the pores. The shape of the pores ranges from flat to spherical to fibre-like, and their orientation is assumed to obey a given probability distribution function. As an example of straightforward physical interpretation, the isotropic case of randomly oriented voids-pores is studied as a function of the matrix Poisson's ratio and the porosity (void volumetric fraction). The results are in good agreement with published experimental data on porous metals, and with previous isotropic solutions in the literature.

1. Introduction

In the research on composite materials with inclusions, a milestone has been set by Eshelby [4], who described the effect, in terms of elastic strain energy, stress, and strain, of the presence of an elastic spheroidal inclusion (i.e., a revolution ellipsoid), in an isotropic, linear elastic, infinite matrix. The key quantity in Eshelby's approach is a fourth-order tensor that was called by other authors Eshelby's tensor, depending on the Poisson's ratio of the matrix and the aspect ratio of the inclusion, the ratio of the semi-axis in the direction of the symmetry axis to the semi-axis in the transverse plane.

The self-consistent method of Hill [7], for two-phase composites with matrix and particulate, and the heuristic method of Budiansky [1], for several phases of isotropic inclusions of spherical shape, give results equivalent to those of Eshelby's theory [4], in their respective limits of applicability.

Based on the use of Eshelby's tensor, Walpole [18–21] developed a homogenisation method for composites with an isotropic matrix and N inclusion phases, each constituted by anisotropic inclusions with a given aspect ratio and aligned in the same direction. Mori and Tanaka [12] made use of Eshelby's tensor to calculate the average stress in the matrix and the average elastic energy in metals with misfitting precipitates.

Weng [23] reformulated the Mori–Tanaka method [12] along the lines of Walpole's work [18–20], and showed that the two approaches are in fact equivalent. Subsequently, Qiu and Weng [13] used the reformulated Mori–Tanaka method to calculate the effective elastic moduli of some cases of aligned and randomly oriented inclusions. Weng's group has also applied Eshelby's theory [4] to composites with an elastoplastic matrix and elastic inclusions [17,14], and elastoplastic matrix with voids [15].

In a previous work [6], we generalised Walpole's method to the case in which the orientation of the inclusions in one of the N inclusion families obeys a given probability distribution; this includes the cases of randomly oriented inclusions and of aligned inclusions as particular cases.

Here, following the idea that porous materials can be seen as a particular case of inclusions with zero elastic stiffness, i.e., voids, the aim is to obtain the elasticity tensor of a porous material given the elasticity tensor of the solid matrix, and the directional arrangement and aspect ratio of the voids. The proposed model is constructed in steps.

First, the previously developed homogenisation procedure [6] is extended to the case in which the orientation of each of the N families of spheroidal inclusions obeys a probability distribution. Second, the number of inclusion families passes from discrete (N) to a continuous infinity, parameterised by the aspect ratio, varying from zero (flat discs) to one (spheres) to infinity (needles-fibres). Third, the equations are specialised to the case in which the inclusions are voids, i.e., their intrinsic elasticity tensor vanishes identically.

The isotropic case, attained when the inclusion orientation is random, is studied for its simplicity. The bulk modulus, the shear modulus, and the Poisson's ratio of the porous material are evaluated for two sample aspect ratio probability distributions

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and for values of the matrix Poisson's ratio between zero and 1/2 (incompressible matrix). For increasing matrix Poisson's ratio, the bulk and shear moduli decrease and increase, respectively. In contrast, the choice of the aspect ratio probability distribution does not change the results significantly. As a comparison, the ideal case of no matrix-void interaction is also studied: the moduli are linearly rescaled by the solid volumetric fraction, which results in a consistent overestimation, with respect to the proposed method.

In its isotropic version, the proposed model gives very good agreement with the experimental results obtained by Spitzig et al. [16], and with the predictions of the application of the models of Hill [7] and Budiansky [1] to porous materials, which are reported by Spitzig et al. [16].

With respect to Hill's [7] and Budiansky's [1] models, as well as the earlier model by MacKenzie [11], the proposed model has the advantage of being able to account for any material symmetry and inclusion (in this case pore/void) orientation.

In the reminder of this section, some basics are given about the notation and the algebra of isotropic and transversely isotropic fourth-order tensors, along with the presentation of Walpole's method [20,21] for composites with aligned inclusions.

1.1. Basic notation

Since this work is developed within the framework of linear elasticity, i.e., infinitesimal deformations, we make no distinction between the material and the spatial pictures of Mechanics, which are often distinguished by the use of uppercase variables for the former, and lower case or small-caps variables for the latter.

The second-order identity tensor is denoted $\boldsymbol{I}$, with components $I_{ij} = \delta_{ij}$. The fourth-order identity tensor is obtained by means of the special tensor products $\overline{\otimes}$ and $\underline{\otimes}$ as defined by Curnier [3], involving the second-order identity $\boldsymbol{I}$:

$$\mathbb{I} = \tfrac{1}{2}(\boldsymbol{I}\overline{\otimes}\boldsymbol{I} + \boldsymbol{I}\underline{\otimes}\boldsymbol{I}), \quad I_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{1}$$

A fourth-order tensor $\mathbb{T}$ is endowed with pair symmetry if its components with respect to any basis $\{\boldsymbol{e}\}_{i=1}^{3}$ are such that $T_{ijkl} = T_{jikl} = T_{ijlk} = T_{jilk}$ and with diagonal symmetry if $T_{ijkl} = T_{klij}$. When a tensor possesses both symmetries, it is said to be fully symmetric. The identity $\mathbb{I}$ (Eq. (1)) is a fully symmetric tensor. For every fourth-order tensors $\mathbb{T}$ and $\mathbb{Y}$, the "standard" contraction is denoted $\mathbb{T} : \mathbb{Y}$, with $(\mathbb{T} : \mathbb{Y})_{ijkl} = T_{ijpq} Y_{pqkl}$, and the "full" contraction is denoted $\mathbb{T} :: \mathbb{Y} = T_{ijkl} Y_{ijkl}$. The set of all directions in space is described by the unit sphere $\mathbb{S}^2 = \{\boldsymbol{m} \in \mathbb{R}^3 : \|\boldsymbol{m}\| = 1\}$.

1.2. Isotropic fourth-order tensors

Isotropy is the invariance under every rotation. The basis for fourth-order isotropic tensors is obtained by decomposition of the symmetric identity into

$$\mathbb{I} = \mathbb{K} + \mathbb{M}, \tag{2}$$

where

$$\mathbb{K} = \tfrac{1}{3}(\boldsymbol{I} \otimes \boldsymbol{I}), \quad K_{ijkl} = \tfrac{1}{3}\delta_{ij}\delta_{kl}, \tag{3a}$$

$$\mathbb{M} = \mathbb{I} - \mathbb{K}, \quad M_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \tfrac{1}{3}\delta_{ij}\delta_{kl}, \tag{3b}$$

are the fully symmetric spherical and deviatoric operators, respectively (see, e.g., [21,5]), which are orthogonal (in the sense that $\mathbb{K} : \mathbb{M} = \mathbb{M} : \mathbb{K} = \mathbb{O}$) and idempotent (i.e., $\mathbb{K} : \mathbb{K} = \mathbb{K}$ and $\mathbb{M} : \mathbb{M} = \mathbb{M}$).

Any isotropic fully symmetric fourth-order tensor $\mathbb{Q}$ can be written as the linear combination of $\mathbb{K}$ and $\mathbb{M}$, and its components are found by means of the contractions

$$Q^K = \mathbb{Q} :: \mathbb{K} = \tfrac{1}{3} Q_{iijj}, \tag{4a}$$

$$Q^M = \tfrac{1}{5}\,\mathbb{Q} :: \mathbb{M} = \tfrac{1}{5}(Q_{ijij} - \tfrac{1}{3} Q_{iijj}). \tag{4b}$$

Due to the orthogonality of $\mathbb{K}$ and $\mathbb{M}$, the product of two isotropic tensors and the inverse of an isotropic tensor are fully decoupled in the two components. Therefore, we have

$$\mathbb{Q} : \mathbb{P} = Q^K P^K \mathbb{K} + Q^M P^M \mathbb{M}, \tag{5a}$$

$$\mathbb{Q}^{-1} = (Q^K)^{-1}\mathbb{K} + (Q^M)^{-1}\mathbb{M}. \tag{5b}$$

If $\mathbb{T}$ is any anisotropic tensor, then the contractions (4) give the projection of $\mathbb{T}$ onto the isotropic subspace, which coincide with the isotropic directional average of $\mathbb{T}$ [6]:

$$\langle \mathbb{T} \rangle^K = \mathbb{T} :: \mathbb{K} = \tfrac{1}{3} T_{iijj}, \tag{6a}$$

$$\langle \mathbb{T} \rangle^M = \tfrac{1}{5}\,\mathbb{T} :: \mathbb{M} = \tfrac{1}{5}(T_{ijij} - \tfrac{1}{3} T_{iijj}). \tag{6b}$$

1.3. Transversely isotropic fourth-order tensors

Transverse isotropy is the invariance under rotations about a given direction, represented by the unit vector $\boldsymbol{m} \in \mathbb{S}^2$. A fourth-order tensor $\mathbb{T}$, transversely isotropic with respect to a direction $\boldsymbol{m}$, can be decomposed in Walpole's basis $\{\mathbb{B}_p\}_{p=1}^{6}$ relative to direction $\boldsymbol{m}$ (see Appendix A for details), as

$$\mathbb{T} = \hat{T}^p \mathbb{B}_p. \tag{7}$$

Walpole's components $\hat{T}^p$ of $\mathbb{T}$ can be collected into the vector $\hat{T}$, sometimes called Walpole's vector [6]. It is important to note that a tensor $\mathbb{T}$, transversely isotropic with respect to a direction $\boldsymbol{m}$, expressed as an *explicit function* of $\boldsymbol{m}$, reads

$$\mathbb{T}(\boldsymbol{m}) = \hat{T}^p \mathbb{B}_p(\boldsymbol{m}), \tag{8}$$

i.e., the components are independent of the direction, and the dependence on $\boldsymbol{m}$ is entirely taken by Walpole's basis tensors $\mathbb{B}_p$. The product $\mathbb{Z} = \mathbb{T} : \mathbb{Y}$ of two tensors $\mathbb{T}$ and $\mathbb{Y}$ and the inverse $\mathbb{T}^{-1}$ of a tensor $\mathbb{T}$ are given, in Walpole's components, by

$$\hat{Z} = (\hat{T}^1\hat{Y}^1 + \hat{T}^3\hat{Y}^4, \hat{T}^2\hat{Y}^2 + \hat{T}^4\hat{Y}^3, \hat{T}^1\hat{Y}^3 + \hat{T}^3\hat{Y}^2, \hat{T}^2\hat{Y}^4 + \hat{T}^4\hat{Y}^1, \hat{T}^5\hat{Y}^5, \hat{T}^6\hat{Y}^6), \tag{9a}$$

$$\hat{T}^{-1} = \left(\frac{\hat{T}^2}{\hat{T}^1\hat{T}^2 - \hat{T}^3\hat{Z}^4}, \frac{\hat{T}^1}{\hat{T}^1\hat{T}^2 - \hat{T}^3\hat{Z}^4}, \frac{-\hat{T}^3}{\hat{T}^1\hat{T}^2 - \hat{T}^3\hat{Z}^4}, \frac{-\hat{T}^4}{\hat{T}^1\hat{T}^2 - \hat{T}^3\hat{Z}^4}, \frac{1}{\hat{T}^5}, \frac{1}{\hat{T}^6}\right). \tag{9b}$$

By means of the contractions (6), it can be shown that the isotropic projection of a transversely isotropic tensor has components

$$\langle \mathbb{T} \rangle^K = \tfrac{1}{3}[\hat{T}^1 + 2\hat{T}^2 + \sqrt{2}\hat{T}^3 + \sqrt{2}\hat{T}^4], \tag{10a}$$

$$\langle \mathbb{T} \rangle^M = \tfrac{1}{15}[2\hat{T}^1 + \hat{T}^2 - \sqrt{2}\hat{T}^3 - \sqrt{2}\hat{T}^4 + 6\hat{T}^5 + 6\hat{T}^6]. \tag{10b}$$

1.4. Walpole's homogenisation method for aligned inclusions

Walpole's method [20] allows for the determination of the linear elasticity tensor of a composite constituted by a matrix (index 0) and N phases (or families) of inclusions. The matrix is assumed to be isotropic, and the inclusions in each phase r are assumed to be spheroids (i.e., revolution ellipsoids) aligned in a given direction, and their material properties transversely isotropic in the same direction. Under these hypotheses, the overall