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A circular inhomogeneity subjected to non-uniform remote loading in finite plane elastostatics

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Abstract

We consider an inhomogeneity–matrix system from a particular class of compressible hyperelastic materials of harmonic-type undergoing finite plane deformations. We obtain the complete solution for a perfectly bonded circular inhomogeneity when the system is subjected to non-uniform remote stress characterized by stress functions described by general polynomials of order $n \geqslant 1$ in the corresponding complex-variable z used to describe the matrix.

MSC: 74B20; 30E25; 30B10

Keywords: Elastic inhomogeneity; Finite elasticity; Non-uniform remote loading; Complex-variable methods

1. Introduction

Problems involving elastic inhomogeneities have received a considerable amount of attention in the literature (see, for example, [1] for an extensive bibliography). In many of these cases, complex-variable methods are used extensively and successfully to produce exact solutions and significant practical results in problems of linear plane and anti-plane elastostatics. In contrast, however, such exact analyses have been more or less absent in the analogous problems from *finite* elasticity. This might be attributed to the lack of availability of a comparable (with regard to ease of application) complex-variable formulation.

Recent works by Ogden and Isherwood [2] and Varley and Cumberbatch [3] have provided rather nice complex-variable formulations of a class of problems involving the plane-strain deformations of a set of compressible hyperelastic materials of harmonic-type, originally proposed by John [4]. These materials have attracted considerable attention in the literature recently in both their practical applications and in their theoretical investigation (see, for example, [5–12]). More recently, Ru [13] has developed the complex-variable formulation presented in [3] and obtained a relatively simple version particularly suitable for the study of problems involving elastic inhomogeneities for the same class of harmonic materials. Introductory problems involving elastic inhomogeneities for this class of (harmonic) materials have been studied in [2,14]. The results presented there, however, are limited by the fact that the inhomogeneity–matrix system is subjected exclusively to *uniform* remote loading.

In this paper, we adopt the complex-variable formulation presented in [13] (mainly because of its relatively simple application) and generalize and extend the work begun in [2,14] by considering plane finite deformations of a circular elastic inhomogeneity embedded in the same class of harmonic materials under the assumption of *non-uniform* remote loading. In particular, we obtain the complete solution for a perfectly bonded circular inhomogeneity when the system is subjected to arbitrary remote stress characterized by stress functions described by general polynomials of degree $n \geqslant 1$ in the corresponding complex variable z describing the matrix. The analysis of this class of problems is extremely important in that, essentially, it accommodates a wide range of problems which incorporate more general forms of (inhomogeneous) remote loading.

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2. Notation and prerequisites

Let $z = x_1 + ix_2$ be the initial coordinates of a material particle in the undeformed configuration and $w(z) = y_1(z) + iy_2(z)$, the corresponding spatial coordinates in the deformed configuration. The components of the deformation gradient tensor are given by

$$F_{ij} = \frac{\partial y_i}{\partial x_j} = y_{i,j},$$

and we define the following scalar invariants:

$$I = \lambda_1 + \lambda_2 = \sqrt{F_{ij}F_{ij} + 2J}, \quad J = \lambda_1\lambda_2 = \det[F_{ij}],$$

where λ_1 and λ_2 are principal stretches.

Harmonic materials [4] are characterized by the following strain-energy density W defined per-unit-area of the reference configuration:

$$W = 2\mu[F(I) - J], \tag{1}$$

where μ is a given (positive) material constant and $F(I)$ is a material function of I. To give some insight into the special class of harmonic materials discussed in [3], we consider the case of uniaxial tension. For a harmonic material, the uniaxial Piola stress is given by [3,13]

$$T = 2\mu[F'(I) - \lambda_2].$$

The transverse stretch λ_2 vanishes in this case so that [13]

$$F'(I) = \lambda_1, \quad \lambda_2 = P(\lambda_1) - \lambda_1,$$

where λ_1 is the uniaxial stretch and P denotes the inverse of the function F'. Consequently,

$$T = 2\mu[2\lambda_1 - P(\lambda_1)] \tag{2}$$

and the function P of a harmonic material is determined by its uniaxial relation. Varley and Cumberbatch [3] confined their discussions to a special case of (2) which includes the undeformed state $\lambda_1 = 1$ at which T is required to be zero. In this special case, Ru [13] shows that the uniaxial relation (2) becomes

$$T = 4\mu\frac{b\lambda_1^k + c}{\lambda_1}, \quad k = 2(1+\delta), \tag{3}$$

where δ, b and c are some arbitrary real numbers introduced in [13] as part of Ru's formulation. According to [3], the value $k = 2.28$ provides the best agreement with experimental data obtained from some rubber-like materials. Consequently, the class of harmonic materials (3) defined by $k = 2$ has gained particular attention. Omitting details, Ru [13] shows that this particular class of harmonic materials is characterized by

$$F'(I) = \frac{1}{4\alpha}[I + \sqrt{I^2 - 16\alpha\beta}], \quad P(\lambda) = 2\left(\alpha\lambda + \frac{\beta}{\lambda}\right) \geqslant 4\sqrt{\alpha\beta},$$
$$T(\lambda_1) = 4\mu\left[(1-\alpha)\lambda_1 - \frac{\beta}{\lambda_1}\right], \quad 1 > \alpha \geqslant \frac{1}{2}, \quad \beta > 0, \tag{4}$$

where α and β are two material constants, related to b and c by $\alpha = 1 - b$ and $\beta = -c$. Here, the restriction $1 > \alpha \geqslant \frac{1}{2}$ is required to obtain a positive Piola stress and transverse stretch at very large stretching. Similarly, to get a negative Piola stress at very large compression, we have $\beta > 0$ (see [13] for details). In addition, to incorporate the undeformed stage, we require that, for $\lambda_1 = 1$, $T = 0$. That is,

$$T(1) = 4\mu\left[(1-\alpha)1 - \frac{\beta}{1}\right] = 0,$$

which gives

$$\alpha + \beta = 1. \tag{5}$$