International Journal of Non-Linear Mechanics 82 (2016) 6–16

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Review

On energy wave equations with non-negative non-linear terms

M. Milla Miranda [a], A.T. Louredo [a], M.R. Clark [b], H.R. Clark [c,*]

[a] Universidade Estadual da Paraíba, DM, PB, Brazil
[b] Universidade Federal do Piauí, DM, PI, Brazil
[c] Universidade Federal Fluminense, IME, RJ, Brazil

ARTICLE INFO

Article history:
Received 3 January 2015
Received in revised form
22 December 2015
Accepted 22 December 2015
Available online 27 February 2016

Keywords:
Existence of solutions
Uniform stabilization
Non-linear system with non-negative terms

ABSTRACT

This paper deals with the existence of at least one solution and the uniform stabilization of the energy of an initial-boundary value problem for a non-linear wave equation with non-linear boundary condition of the feedback type. The non-linearities in both waves and boundary equations behave as functions of the type $|u|_{\mathbb{R}}^{\varrho}$ for $\varrho > 1$.

Contents

1. Introduction

Let Ω be an open, bounded and connected set of $\mathbb{R}^n$ with its boundary Γ of class C^2. Suppose also that Γ is partitioned into Γ_0 and Γ_1 both with positive measure and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ empty.

This paper is concerned with the global existence of solutions and uniform stability of the energy for the following non-linear

* Corresponding author. Tel.: +55 21 2629 2060.
E-mail addresses: milla@im.ufrj.br (M. Milla Miranda), aldotl@cct.uepb.edu.br (A.T. Louredo), mclark@ufpi.br (M.R. Clark), hclark@vm.uff.br (H.R. Clark).

http://dx.doi.org/10.1016/j.ijnonlinmec.2015.12.007
0020-7462/

initial-boundary value problem:

$$\left|\begin{array}{ll} u''-\mu\Delta u+g(\cdot,u)=f & \text{in } \Omega\times(0,\infty),\\ u=0 & \text{on } \Gamma_0\times(0,\infty),\\ \dfrac{\partial u}{\partial\nu}+h(\cdot,u')+q(\cdot,u)=0 & \text{on } \Gamma_1\times(0,\infty),\\ u(0)=u^0,\quad u'(0)=u^1 & \text{in } \Omega,\end{array}\right. \tag{1.1}$$

where ν is the unit outward normal on Γ_1, the real-valued functions $\mu=\mu(t)$, $f=f(x,t)$, $g=g(x,s)$, $h=h(x,s)$ and $q=q(x,s)$ are defined in $\mathbb{R}_+$, $\Omega\times\mathbb{R}_+$, $\Omega\times\mathbb{R}$, $\Gamma_1\times\mathbb{R}$ and $\Gamma_1\times\mathbb{R}$, respectively. Moreover, the functions g and q behave as $|s|^{\rho}_{\mathbb{R}}$ and $|s|^{\sigma}_{\mathbb{R}}$, respectively, for $\rho>1$ and $\sigma>1$, and h is a continuous and strong monotone function in the variable s.

1.1. Physical motivation and some previous works

Problem (1.1) comes from many physical situations. Perhaps the most significant of them is the saturation property of the nuclei of atoms. Thus, the non-linear equation $(1.1)_1$ arises in quantum theory of the meson in its classical aspects. In fact, in Schiff's paper [23], it is developed to account for nuclear saturation and shell structure in terms of many-body forces. These forces are derived from mesons that obey the non-linear wave equation

$$\frac{\partial^2 u}{\partial t^2}-\Delta u+G'(u)=fF'(u)\quad \text{in } \Omega\times(0,\infty),$$

where the prime denotes the derivative with respect to $u, f=f(x,t)$ is the nucleon source density, $F(u)$ is the non-linear coupling function and $G(u)$ is the non-linear field function. In particular, a typical physical example of $G(u)$ is an even function, given by $G(u)=\frac{1}{2}u^2+\frac{1}{4}u^4$.

In this context, Jörgens [5], in the absence of sources ($f=0$), formulated the following equation as a version to the motion of quantum mechanics:

$$\frac{\partial^2 u}{\partial t^2}-\Delta u+\mu^2 u+\eta^2|u|^2u=0\quad \text{in } \Omega\times(0,\infty),$$

and proved the existence and uniqueness of solutions for an initial-boundary value problem associated with the above equation. Indeed, in Jörgens [5,6] began a rigorous mathematical research of equations of the type

$$\frac{\partial^2 u}{\partial t^2}-\Delta u+F'(|u|^2_{\mathbb{R}})u=0\quad \text{in } \Omega\times(0,\infty).$$

Later, Lions and Strauss [14] developed a large fields of research on non-linear evolution equations which include the models of Schiff and Jörgens.

In the above papers the non-linearities in the wave equation behave as $|u|^{\rho}u$, and thus the energy method works well. When the non-linearity is for instance of the type u^2, we need, however, a totally different approach. Lions [12] analyzed this using the potential well method, which was introduced by Sattinger in [22]. Tartar [25] developed even another method to study the same non-linearity.

Recently, Medeiros et al. [17] showed via Tartar and energy methods the existence and uniqueness of global solutions for the problem

$$\left|\begin{array}{ll} u''-\Delta u+|u|^{\rho}_{\mathbb{R}}=f & \text{in } \Omega\times(0,\infty),\\ u=0 & \text{on } \Gamma\times(0,\infty),\\ u(x,0)=u^0(x),\quad u'(x,0)=u^1(x) & \text{in } \Omega,\end{array}\right. \tag{1.2}$$

with restrictions on the size of the norms of the initial data u^0 and u^1, and $\rho>1$. Observe that Eq. $(1.2)_1$ is obtained when we set $F(u)\ =u$ and $G(u)=\frac{1}{\rho+1}|u|^{\rho}u$ in Schiff 's model.

Another significant problem in our research is the initial-boundary value problem with both Dirichlet and feedback boundary conditions

$$\left|\begin{array}{ll} u''-\mu\Delta u=0 & \text{in } \Omega\times(0,\infty),\\ u=0 & \text{on } \Gamma_0\times(0,\infty),\\ \mu\dfrac{\partial u}{\partial\nu}+\delta u'=0 & \text{on } \Gamma_1\times(0,\infty),\\ u(x,0)=u^0(x),\quad u'(x,0)=u^1(x) & \text{in } \Omega.\end{array}\right. \tag{1.3}$$

When μ is a constant, Komornik and Zuazua [7], Lasiecka and Triggiani [11], Quinn and Russell [21], among others proved by applying semigroups theory the existence and uniqueness of solutions for system (1.3). This method does not work when in $(1.3)_1$ the coefficient is a time-dependent function, i.e., $\mu=\mu(t)$. In this case, Milla Miranda and Medeiros [18] studied the existence and uniqueness of solutions via Faedo–Galerkin–Lions' method, by constructing a special Hilbertian basis that satisfies the feedback boundary condition at $t=0$. Furthermore, the asymptotic behavior of the energy was also established.

Many non-linear problems with non-linearities in the wave equation and in the boundary equation are related with our problem (1.1). We refer to the reader the works of Komornik [8], Lasiecka and Tataru [10] and Zuazua [29]. There, it is included a qualitative analysis of the properties of the solutions and the solvability of these problems via semigroup theory. In addition to these works, we cite Cavalcanti et al. [4], Milla Miranda and San Gil Jutuca [19], and Vitillaro [26]. All these last authors established existence of solutions via Faedo–Galerkin–Lions' method, and some qualitative properties of the solutions.

The proposed problem (1.1) generalizes substantially problems (1.2) and (1.3), since all equations in (1.1) are non-linear. This means that problem (1.1) is physically more realistic, and as a consequence, we must deal with more technical difficulties.

In significant works of Lasiecka and Tataru [10] and Cavalcanti et al. [3], the existence of solutions of similar problems to (1.1) is investigated. In [3] the main ideas of [10] are used and they obtained the existence of global solutions. In our paper, the method used to obtain the existence of global solutions of (1.1) is different from that used in [3], and consequently in [10], because there is no a priori control of the sign of some terms of the energy.

As we are about to see, our strategy to overcome this problem is to use an idea introduced by Tartar [25]. Moreover our boundary dissipation $h(\cdot,u')$ depends on x, and one part of the potential energy of system (1.1) depends on t. This makes more difficult to obtain a solution for problem (1.1).

One way to obtain the blow-up of the solutions in finite time for problem (1.1) is to have hypotheses that let you know the sign of the source terms $g(\cdot,u)$ and $q(\cdot,u)$. Note that these terms are related with

$$\int_\Omega G(\cdot,u)\,dx\quad\text{and}\quad\int_{\Gamma_1}Q(\cdot,u)\,d\Gamma$$

which constitute a part of the energy of system (1.1). Here $G(x,s)$ and $Q(x,s)$ denote the anti-derivative with respect to s of the functions $g(x,s)$ and $q(x,s)$, respectively. In our case, we have no information on the sign because $g(x,s)$ and $q(x,s)$ behave like $\alpha(x)|s|^{\rho}$ and $\beta(x)|s|^{\rho}$, respectively, with $\alpha\in L^\infty(\Omega)$ and $\beta\in L^\infty(\Gamma_1)$. In fact, if $g(x,s)=|s|^{\rho}$ and $q(x,s)=|s|^{\sigma}$, then the expressions

$$\int_\Omega G(\cdot,u)\,dx=\frac{1}{\rho+1}\int_\Omega|u|^{\rho}u\,dx\quad\text{and}\quad\int_{\Gamma_1}Q(\cdot,u)\,dx=\frac{1}{\sigma+1}\int_\Omega|u|^{\sigma}u\,d\Gamma$$

do not have the same sign for all t. Consequently, the blow-up properties of the energy in finite time are an open question. However, see Cavalcanti et al. [3], if $g(s)=|s|^{\rho}s$ and $q(s)=|s|^{\sigma}s$, then the blow-up question is answered because the signs of