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Interaction of an acceleration wave with a characteristic shock in a non-ideal relaxing gas

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ABSTRACT

The amplitude of an acceleration wave propagating along the characteristic associated with the largest eigenvalue in a non-ideal relaxing gas is evaluated. The evolution of a characteristic shock and its interaction with the acceleration wave is studied. The amplitudes of the reflected and transmitted waves and the jump in the shock wave acceleration after interaction are computed. The effects of relaxation and non-ideality on the amplitude of acceleration wave are discussed.

1. Introduction

The term acceleration wave means an isolated geometric surface that moves relative to the material and across which the acceleration (but not the velocity) is discontinuous [1]. In Continuum Mechanics, the acceleration waves are also known as weak discontinuity waves (i.e. $C^{(1)}$ waves) and are important kind of solutions of non-linear hyperbolic systems. These waves are characterized by a discontinuity in a normal derivative of the field but not in the field itself [2]. The evolution of a weak discontinuity or acceleration wave for a hyperbolic quasi-linear system of equations satisfying Bernoulli's law is described extensively in the literatures [3–5].

For a characteristic shock, the shock surface coincides with a characteristic surface and its velocity with an eigenvalue of the system, both ahead and behind the shock. The corresponding eigenvalue may be single or have multiplicity $j>1$. In the case where the eigenvalue is single, the shock is characteristic if and only if the exceptionality condition $\nabla\lambda \cdot R = 0$ is satisfied, where λ is the eigenvalue (and also the velocity of the characteristic shock), R is the corresponding right eigenvector of the system and ∇ is the gradient with respect to the field vector [6,7]. The shock corresponding to the multiple eigenvalue that has multiplicity $j>1$ is always exceptional [8–10].

The works of Jeffrey [11] and Boillat and Ruggeri [12] are the origin of the general theory of wave interactions. The shock undergoes an acceleration jump as a consequence of an interaction with a weak wave [12–14]. Radha et al. [15] verified that the general theory of wave interaction problem which originated from the work of Jeffrey [11] leads to the results obtained by Brun [14] and Boillat and Ruggeri [12]. The theory has been successfully applied to study the interaction of discontinuity wave with a characteristic shock or a strong shock in the mediums like shallow water, relaxing gas, dusty gas, transient pinched plasma and non-ideal gas [16–23].

At high temperatures, the internal energy of the gas molecules consists of translational, rotational and vibrational components. When a gas is compressed by a receding piston or by the passage of a shock front, the whole energy goes initially to increase the translational energy, and it is followed by a relaxation from translational mode to rotational mode and also from translational mode to vibrational mode until the equilibrium between these modes is re-established [24]. The process is called relaxation and the departure from equilibrium is due to vibrational relaxation; the rotational and translational modes are assumed to be in local thermodynamical equilibrium throughout. Arora et al. [25] used a similarity method to study imploding strong shocks in a non-ideal relaxing gas with van der Waals equation of state. One dimensional steepening of waves in non-ideal relaxing gas is studied in [26] and it is observed that the transport equations for the discontinuities in the first order derivatives of the flow variables lead to Bernoulli type of equations.

In this paper, we considered a system of partial differential equations describing the one dimensional unsteady plane and radially symmetric flow of an inviscid vibrationally relaxing gas with van der Waals equation of state. The evolution of a characteristic shock is studied and the amplitude of the acceleration

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wave propagating along the characteristic associated with the largest eigenvalue is evaluated. The interaction of the acceleration wave with the characteristic shock is considered and the jump in the shock acceleration and the amplitudes of reflected and transmitted waves after interaction are evaluated by using the results of general theory of wave interaction [15].

2. Basic equations

We consider a system of partial differential equations describing the one dimensional unsteady plane ($m=0$), cylindrically symmetric ($m=1$) and spherically symmetric ($m=2$) flow of an inviscid vibrationally relaxing gas with van der Waals equation of state as [25]

$$U_t + AU_x = f, \tag{1}$$

where $U=(\rho,u,p,\sigma)^{tr}$, $f=(-m\rho u/x, 0, -\gamma pmu/((1-b\rho)x)-(\gamma-1)\rho Q, Q)^{tr}$ and

$$A=\begin{pmatrix} u & \rho & 0 & 0\\ 0 & u & 1/\rho & 0\\ 0 & \gamma p/(1-b\rho) & u & 0\\ 0 & 0 & 0 & u\end{pmatrix}.$$

Here, x is the distance, t the time, ρ the density, u the particle velocity, p the pressure, σ the vibrational energy, γ the ratio of specific heats, b the van der Waals excluded volume and lies in the range $0.9\times 10^{-3}\le b\le 1.1\times 10^{-3}$ (SI unit of b is m^3/kg) [27,28]. It may ne noted that $b=0$ corresponds to the case of ideal relaxing gas [24]. The quantity Q is the rate of change of vibrational energy, and is a function of p, ρ and σ, given by

$$Q=\frac{(\overline{\sigma}(p,\rho)-\sigma)}{\tau},$$

where $\overline{\sigma}=\sigma_e+c\{p(1-b\rho)/\rho-(1-b\rho_e)p_e/\rho_e\}$ is the equilibrium value of σ and the suffix e refers to an initial equilibrium reference state; the quantities τ and c are the relaxation time and the ratio of vibrational specific heat to the specific gas constant, respectively. If not stated otherwise, a variable as a subscript indicates partial differentiation with respect to that variable.

The van der Waals equation of state is of the form

$$p=\frac{\rho RT}{(1-b\rho)},$$

where R is the specific gas constant and T is the translational temperature.

The matrix A in Eq. (1) has eigenvalues

$$\lambda^{(1)}=u+a,\quad \lambda^{(2,1)}=\lambda^{(2,2)}=u\ (\lambda^{(2)}=u \text{ is a double root}),\quad \lambda^{(3)}=u-a, \tag{2}$$

with the corresponding left and right eigenvectors

$$\begin{aligned}
&L^{(1)}=(0,\rho a,1,0),\quad R^{(1)}=(1/(2a^2),1/(2\rho a),1/2,0)^{tr},\\
&L^{(2,1)}=(-a^2,0,1,0),\quad R^{(2,1)}=(-1/a^2,0,0,0)^{tr},\\
&L^{(2,2)}=(0,0,0,1),\quad R^{(2,2)}=(0,0,0,1)^{tr},\\
&L^{(3)}=(0,-\rho a,1,0),\quad R^{(3)}=(1/(2a^2),-1/(2\rho a),1/2,0)^{tr},
\end{aligned} \tag{3}$$

where $a=\left(\frac{\gamma p}{\rho(1-b\rho)}\right)^{\frac{1}{2}}$ is the frozen speed of sound. Since, the multiplicity of the eigenvalue $\lambda^{(2)}=u$ is 2, there exists a characteristic shock propagating with the speed $V=u$. Using the fact that across a characteristic shock, no mass flow takes place, the Rankine–Hugoniot conditions across this shock are given by $[u]=0$, $[p]=0$, $[\rho]=\zeta$, $[\sigma]=\eta$ where ζ and η are functions of t. Here, $[X]=X-X_*$ denotes the jump in X across the characteristic shock where X_* and X are the values just ahead of the shock and behind the shock, respectively. Multiplying (1) by eigenvectors $L^{(2,1)}$ and $L^{(2,2)}$, respectively, and then on forming the jumps across the characteristic shock, we get the evolutionary law for ζ and η

$$L\frac{d[U]}{dt}+[L]\frac{dU_*}{dt}=L[f]+[L]f_*, \tag{4}$$

where $\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}$ denotes the material derivative following the shock. Now, using (1) and (3) in (4) we obtain the following transport equations for the quantities ζ and η

$$\begin{aligned}
\frac{d\zeta}{dt}&=-\zeta\left(u_x+\frac{mu}{x}\right)\left(\frac{1-b(2\rho-\zeta)}{1-b\rho}\right)+\frac{(\gamma-1)(\rho-\zeta)(1-b(\rho-\zeta))}{\gamma p\tau}\\
&\quad\times\left\{\zeta\left(\sigma_0-\frac{cp_0(1-b\rho_0)}{\rho_0}-\sigma+\eta-cpb\right)-\rho\eta\right\},\\
\frac{d\eta}{dt}&=-\frac{1}{\tau}\left(\eta+\frac{cp\zeta(1-2b\rho)}{\rho(\rho-\zeta)}\right).
\end{aligned} \tag{5}$$

2.1. Particular case

Let us consider the case exhibiting the space–time dependence when the flow variables u,ρ,p,σ behind the characteristic shock are given by

$$u(x,t)=k(t)x,\quad \rho=\rho(t),\quad p=p(t),\quad \sigma=\sigma(t). \tag{6}$$

In this case the particle velocity exhibits linear dependence on x and such a state can be visualized in terms of an atmosphere filled with a gas which has spatially uniform pressure variation on account of the particle motion and the spatially uniform relaxation rate [29–31]. This type of velocity distribution is useful in modelling the free expansion of polytropic gasses [29].

Using (6) in equations $(1)_1$ and $(1)_2$ we get the following forms of flow parameters:

$$\rho=\rho_0(1+(t-t_0)k_0)^{-(m+1)},\quad k(t)=k_0/(1+(t-t_0)k_0), \tag{7}$$

whereas equations $(1)_3$ and $(1)_4$ lead to the following system of ordinary differential equations in p and σ

$$\begin{aligned}
&\frac{dp}{dt}+\left(\frac{\gamma(m+1)k}{1-b\rho}+\frac{(\gamma-1)c(1-b\rho)}{\tau}\right)p+\frac{(\gamma-1)\rho}{\tau}\left(\sigma_0-\sigma-\frac{cp_0(1-b\rho_0)}{\rho_0}\right)=0,\\
&\frac{d\sigma}{dt}-\frac{cp(1-b\rho)}{\rho\tau}+\frac{1}{\tau}\left(\sigma-\sigma_0+\frac{cp_0(1-b\rho_0)}{\rho_0}\right)=0,
\end{aligned} \tag{8}$$

where k_0 and ρ_0 corresponds to the initial reference state. Using the dimensionless variables

$$\begin{aligned}
&\tilde{\zeta}=\zeta/\rho_0,\quad \tilde{\rho}=\rho/\rho_0,\quad \tilde{p}=p/p_0,\quad \tilde{\eta}=\eta\rho_0/p_0,\quad \tilde{\sigma}=\sigma\rho_0/p_0,\\
&\tilde{t}=t/t_0,\quad \tilde{\tau}=\tau/t_0,\quad \tilde{k}=kt_0,\quad \tilde{k_0}=k_0t_0,\quad \tilde{b}=b\rho_0,
\end{aligned} \tag{9}$$

and then suppressing the tilde sign, we can write Eq. (5) in the following form:

$$\begin{aligned}
\frac{d\zeta}{dt}&=-(m+1)k\zeta\left(\frac{1-b(2\rho-\zeta)}{1-b\rho}\right)+(\gamma-1)(\rho-\zeta)(1-b(\rho-\zeta))\\
&\quad\times\frac{(\zeta(\sigma_0-c(1-b)-\sigma+\eta-cpb)-\rho\eta)}{(\gamma p\tau)},\\
\frac{d\eta}{dt}&=-\frac{1}{\tau}\left(\eta+\frac{cp\zeta(1-2b\rho)}{\rho(\rho-\zeta)}\right).
\end{aligned} \tag{10}$$

Also, Eqs. (8) in dimensionless form are given by

$$\begin{aligned}
&\frac{dp}{dt}+\left(\frac{\gamma(m+1)k}{1-b\rho}+\frac{(\gamma-1)c(1-b\rho)}{\tau}\right)p+\frac{(\gamma-1)\rho(\sigma_0-c(1-b)-\sigma)}{\tau}=0,\\
&\frac{d\sigma}{dt}-\frac{cp(1-b\rho)}{\rho\tau}+\frac{1}{\tau}(\sigma-\sigma_0+c(1-b))=0,
\end{aligned} \tag{11}$$

where $\rho=(k_0(t-1)+1)^{-(m+1)}$, $k=k_0/(k_0(t-1)+1)$, with $p=p_0$, $\sigma=\sigma_0$, $\zeta=\zeta_0$ and $\eta=\eta_0$ as the initial conditions as at $t=1$.