International Journal of Non-Linear Mechanics 82 (2016) 69–74

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Constitutive model for third harmonic generation in elastic solids

Vamshi Krishna Chillara [a,*], Cliff J. Lissenden [b]

[a] Acoustics and Sensors group, MPA 11, Los Alamos National Laboratory, NM 87545, United States
[b] Department of Engineering Science and Mechanics, The Pennsylvania State University, 16802 PA, United States

ARTICLE INFO

Article history:
Received 14 August 2015
Received in revised form
23 February 2016
Accepted 24 February 2016
Available online 15 March 2016

Keywords:
Nonlinear elasticity
Nonlinear ultrasonics
Higher harmonic
Generation
Constitutive model

ABSTRACT

In this article, we present a new constitutive model for studying ultrasonic third harmonic generation in elastic solids. The model is hyperelastic in nature with two parameters characterizing the linear elastic material response and two other parameters characterizing the nonlinear response. The limiting response of the model as the nonlinearity parameters tend to zero is shown to be the well-known St Venant–Kirchhoff model. Also, the symmetric response of the model in tension and compression and its role in third harmonic generation is shown. Numerical simulations are carried out to study third harmonic generation in materials characterized by the proposed constitutive model. Predicted third harmonic guided wave generation reveals an increasing third harmonic content with increasing nonlinearity. On the other hand, the second harmonics are independent of the nonlinearity parameters and are generated due to the geometric nonlinearity. The feasibility of determining the nonlinearity parameters from third harmonic measurements is qualitatively discussed.

1. Introduction

Nonlinear ultrasonic methodologies [1] for damage detection are widely being researched with the primary aim of being able to localize and characterize precursors to macro-scale damage. These methodologies employ ultrasound to probe nonlinearity in the material response to characterize the damage. Several techniques like higher harmonic generation [2], Nonlinear elastic wave spectroscopy (NEWS) [3–5] and Nonlinear resonant ultrasound spectroscopy (NRUS) [6] are developed to detect and characterize damage not easily discernible with conventional ultrasonic methodologies. Of these, ultrasonic higher harmonic generation refers to the generation of higher harmonic frequency components from the primary wave propagating in the material. This generation of higher harmonic frequency components caused by the nonlinear material behavior due to the presence of micro-scale damage is used to decipher the extent of damage progression in the material. Second harmonic generation [7] is widely employed to characterize micro-scale damage in materials, especially metals subject to degradation from fatigue [2], creep [8], radiation damage [9], etc. Many such investigations use bulk-waves that travel in unbounded media to study ultrasonic higher harmonic generation both from theoretical and experimental standpoints. However, there is an increasing interest in the use of nonlinear ultrasonic guided waves [10] and surface waves for early damage detection.

Theoretical investigations concerning the study of nonlinear guided wave propagation in plates [11–13] and pipes [14,15] have been carried out by several researchers. Likewise, numerical studies [16,17] concerning nonlinear guided wave propagation were carried out. However, it should be recognized that the constitutive model used for modeling the nonlinear material behavior in isotropic materials is the same and given by one of the following two equivalent forms of the elastic strain energy function:

1. *Landau–Lifshitz Model*:

$$\mathrm{W}(\mathbf{E})=\tfrac{1}{2}\lambda(\mathrm{tr}(\mathbf{E}))^2+\mu\,\mathrm{tr}(\mathbf{E}^2)+\tfrac{1}{3}C(\mathrm{tr}(\mathbf{E}))^3+B\,\mathrm{tr}(\mathbf{E})\mathrm{tr}(\mathbf{E}^2)+\tfrac{1}{3}A\,\mathrm{tr}(\mathbf{E}^3) \quad (1)$$

2. *Murnaghan Model*:

$$\mathrm{W}(\mathbf{E})=\tfrac{1}{2}\lambda(\mathrm{tr}(\mathbf{E}))^2+\mu\,\mathrm{tr}(\mathbf{E}^2)+\tfrac{1}{3}(l+2m)(\mathrm{tr}(\mathbf{E}))^3-m\,\mathrm{tr}(\mathbf{E})((\mathrm{tr}(\mathbf{E}))^2-\mathrm{tr}(\mathbf{E}^2))+n\det(\mathbf{E}) \quad (2)$$

Here, $\mathbf{E}$ denotes the Lagrangian strain, λ,μ are Lame's constants while A,B,C are called the third order elastic constants and l,m,n are called the Murnaghan constants. The constants (A,B,C) and (l,m,n) are related by $l=B+C$, $m=\frac{1}{2}A+B$ and $n=A$ [18]. Both of these models can be regarded as Taylor series expansions (up to

* Corresponding author.
E-mail address: vchillara@lanl.gov (V.K. Chillara).

http://dx.doi.org/10.1016/j.ijnonlinmec.2016.02.008
0020-7462/

third order) of the strain energy function for an isotropic material about the reference state $\mathbf{E}=0$. In the context of nonlinear ultrasonics, the constants (A,B,C) and (l,m,n) can be interpreted as state variables quantifying the extent of nonlinearity in the material behavior [19]. It is easy to see that $A=B=C=0$ corresponds to linear elastic material behavior with geometric nonlinearity. It should be emphasized that the above models possess two main drawbacks with regard to studying nonlinear wave propagation (both bulk and guided waves) which is of interest to nondestructive evaluation and characterization of materials. They are:

1. Due to the third order nature of the Taylor series expansion, the model can predict only second harmonic generation in materials for primary stress waves having amplitudes of few MPa. Even in cases with high primary wave amplitudes, second harmonic content dominates third harmonic generation.
2. Also, the three independent higher order constants (A,B,C) or (l,m,n) cannot all be determined from higher harmonic generation measurements as there are only two types of waves namely longitudinal and shear waves propagating in the materials.

In fact some of the recent experimental studies [20,21] report the generation of third harmonics in materials and their use to characterize material degradation. In this regard, it is worthwhile to consider developing new models especially for modeling third harmonic generation in elastic solids. To that end, this article presents a new constitutive model for studying third harmonic generation in elastic solids. First, some important aspects of the constitutive model development along with the salient features of the constitutive model are discussed. Then we present some numerical studies pertaining to third harmonic generation in materials characterized by the new constitutive model. Finally, conclusions are drawn.

2. Constitutive model

In this section we present the constitutive model and discuss some important aspects of material behavior depicted by the model with regard to nonlinear wave propagation. But first we introduce the notation used in the section. We denote the deformation gradient by $\mathbf{F}$, the displacement gradient by $\mathbf{H}$ and the Lagrangian strain by $\mathbf{E}$. The following relations exist between these kinematic variables, where $\mathbf{I}$ denotes the Identity tensor.

$$\mathbf{F}=\mathbf{I}+\mathbf{H} \tag{3}$$

$$\mathbf{E}=\tfrac{1}{2}(\mathbf{F}^{\mathbf{T}}\mathbf{F}-\mathbf{I})=\tfrac{1}{2}(\mathbf{H}+\mathbf{H}^{\mathbf{T}}+\mathbf{H}^{\mathbf{T}}\mathbf{H}). \tag{4}$$

The second Piola–Kirchhoff stress is denoted by $(\mathbf{T_{RR}})$. For hyperelastic materials, we denote the strain energy density by W and we have

$$\mathbf{T_{RR}}=\frac{\partial W}{\partial \mathbf{E}} \tag{5}$$

The constitutive model we propose for studying third harmonic generation in isotropic materials is given by the strain energy function

$$W(\mathbf{E})=\frac{1}{2\alpha_1}\left(e^{\left(\alpha_1\lambda(\mathrm{tr}(\mathbf{E}))^2\right)}-1\right)+\frac{1}{\alpha_2}\left(e^{\left(\alpha_2\mu\ \mathrm{tr}(\mathbf{E}^2)\right)}-1\right) \tag{6}$$

Here, λ and μ are Lame's constants and α_1 and α_2 are the nonlinearity parameters as will be evident from the discussion to follow. The following observations need to be made regarding the analytical structure of the proposed model (Eq. (6)):

1. Unlike Eqs. (1) and (2), the above strain energy function is written just in terms of two invariants i.e., $\mathrm{tr}(\mathbf{E})$ and $\mathrm{tr}(\mathbf{E}^2)$.
2. The strain energy function is not an explicit Taylor's expansions i.e., it is not in the polynomial form. However, some interesting insights can be obtained by considering the Taylor expansion of the terms in Eq. (6). We illustrate this by considering the Taylor expansion of the terms in Eq. (6) i.e.,

$$\frac{1}{2\alpha_1}\left(e^{\left(\alpha_1\lambda(\mathrm{tr}(\mathbf{E}))^2\right)}-1\right)=\frac{1}{2}\lambda(\mathrm{tr}(\mathbf{E}))^2+\frac{1}{2}\frac{\alpha_1}{2!}\left(\lambda(\mathrm{tr}(\mathbf{E}))^2\right)^2+\frac{1}{2}\frac{\alpha_1^2}{3!}\left(\lambda(\mathrm{tr}(\mathbf{E}))^2\right)^3+\cdots$$

$$\frac{1}{\alpha_2}\left(e^{\left(\alpha_2\mu\mathrm{tr}(\mathbf{E}^2)\right)}-1\right)=\mu\mathrm{tr}(\mathbf{E}^2)+\frac{\alpha_2}{2!}\left(\mu\mathrm{tr}(\mathbf{E}^2)\right)^2+\frac{\alpha_2^2}{3!}\left(\mu\mathrm{tr}(\mathbf{E}^2)\right)^3+\cdots.$$

The first term in each of the above equations corresponds to the linear elastic material behavior independent of α_1 and α_2. Second terms in the expansion are fourth order in $\mathbf{E}$ and are responsible for third harmonic generation. Likewise, the third term is sixth order in $\mathbf{E}$ and is responsible for fifth harmonic generation, and so on for the other terms. So, the proposed model can in fact predict all the odd harmonics generated due to the material nonlinearity. However, our interest is mainly in third harmonic generation.

The corresponding second Piola–Kirchhoff stress for the proposed model obtained from Eq. (5) is given by

$$\mathbf{T_{RR}}=\lambda\ \mathrm{tr}(\mathbf{E})\left(e^{\left(\alpha_1\lambda(\mathrm{tr}(\mathbf{E}))^2\right)}\right)\mathbf{I}+2\mu\left(e^{\left(\alpha_2\mu\mathrm{tr}(\mathbf{E}^2)\right)}\right)\mathbf{E}. \tag{7}$$

Next, we discuss some important features of the proposed model (Eq. (6)).

3. Features of the proposed model

The proposed model has the following important features.

1. *St Venant–Kirchhoff model as the limiting case*: As mentioned earlier, the nonlinearity parameters in the constitutive model (Eq. (6)) represent the extent of micro-scale damage in the context of nonlinear ultrasonics. Hence, the model is constructed under the restriction that the material response is linear elastic (includes geometric nonlinearity) when the nonlinearity parameters $\alpha_1\to 0$ and $\alpha_2\to 0$. In this case, the limiting strain energy function (from Eq. (6)) as $\alpha_1\to 0$ and $\alpha_2\to 0$ is given by

$$W_{lin}(\mathbf{E})=\tfrac{1}{2}\lambda(\mathrm{tr}(\mathbf{E}))^2+\mu\ \mathrm{tr}(\mathbf{E}^2). \tag{8}$$

This is the well known St Venant–Kirchhoff model in nonlinear hyperelasticity.
2. *Symmetric Tension–Compression response*: It was discussed in [3,19,22] that the tension–compression asymmetry in the material response is responsible for even harmonic generation in elastic materials. In addition, a detailed analysis of such asymmetric response depicted by Eqs. (1) and (2) was discussed in [19,22]. Since we are interested in modeling third (odd) harmonic generation in elastic materials, the proposed model (Eq. (6)) is conceived under the restriction that its response is symmetric in tension and compression. For the present discussion this can be stated as $W(\mathbf{E})=W(-\mathbf{E})$. Next, we discuss the material response depicted by the constitutive model in Eq. (6). We consider the uniaxial stretch deformation given by

$$x_1=sX_1;\quad x_2=X_2;\quad x_3=X_3$$

where $\{x_i\}_{i=1}^3$ denote the coordinates in the current configuration and $\{X_i\}_{i=1}^3$ denote the coordinates in the reference