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Parametric disorder effects on a subcritical stationary bifurcation

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ABSTRACT

For a spatial modulation of the control parameter which describes, for instance, major effects of a rough container boundary in Rayleigh–Bénard convection, the threshold value of the bifurcation from a homogeneous basic state to a spatially periodic state is provided analytically and numerically, taking the one-dimensional cubic–quintic complex Ginzburg–Landau equation with real coefficients as an example. Above the threshold, using the Poincaré–Lindstedt expansion, we show that the quintic term affects both the stationary nonlinear solution and the Nusselt number.

1. Introduction

Nonequilibrium processes often lead by nature to the formation of spatial periodic structures developed from a homogeneous state through the spontaneous breaking of symmetries present in the system [1,2]. The discovery of these localized patterns or localized structures in different experiments such as liquid crystals [3], gas discharge systems [4], chemical reactions [5], fluids [6], granular media [7], and nonlinear optics [8] has been a motivation for theoretical work on localized solutions of amplitude equations [2]. The study of amplitude equations, which can be derived in the vicinity of symmetry-breaking instabilities, has been useful in order to gain an insight into nonequilibrium phenomena in spatially extended systems [2]. As an example, it is well-known that reaction–diffusion, formulated in terms of parabolic partial differential equations, in the vicinity of a supercritical Hopf bifurcation, are described by the following cubic complex Ginzburg–Landau (CGL) equation [2]

$$\tau_0\left(\partial_{\tilde{t}}\tilde{A} - \vec{v}_g.\partial_{\tilde{x}}\tilde{A}\right) = \varepsilon(1+ia)\tilde{A} + \xi_0^2(1+ib)\partial_{\tilde{x}}^2\tilde{A} - g(1+ic)|\tilde{A}|^2\tilde{A}, \qquad (1)$$

where τ_0 is the relaxation time, $\tilde{A}$ is the complex function of time $\tilde{t}$ and space $\tilde{x}$, $\vec{v}_g$ is the linear group velocity, ε measures in a dimensionless scale the distance from the threshold of the instability, i.e., $\varepsilon = (R - R_c)/R_c$, with R being the control parameter that carries the system through the threshold at R_c and ξ_0 is the coherence length. The nonlinear cubic coefficient g determines the amplitude of the pattern as a function of the control parameter ε, the real parameters b and c characterize linear and nonlinear cubic dispersions and $\varepsilon a/\tau_0$ is a correction to the Hopf frequency.

In the case of a supercritical bifurcation, higher-order nonlinearities in Eq. (1) can then be neglected sufficiently near the threshold. If the nonlinear term in Eq. (1) has the opposite sign, which corresponds to a subcritical bifurcation, higher-order nonlinear terms are usually essential. For example, one needs quintic terms to saturate the explosive instabilities provided by the cubic term. Then the cubic–quintic CGL equation, which describes the large-scale modulations of the bifurcated solutions in the vicinity of a strongly subcritical Hopf bifurcation can be written as [2]

$$\tau_0\left(\partial_{\tilde{t}}\tilde{A} - \vec{v}_g.\partial_{\tilde{x}}\tilde{A}\right) = \varepsilon(1+ia)\tilde{A} + \xi_0^2(1+ib)\partial_{\tilde{x}}^2\tilde{A} - g(1+ic)|\tilde{A}|^2\tilde{A} - \gamma(1+id)|\tilde{A}|^4\tilde{A}, \qquad (2)$$

where γ is the nonlinear quintic term and d the nonlinear quintic dispersion term. The other terms have been defined in Eq. (1). For instance, the cubic–quintic CGL equation has been analysed as a model equation to explain the behavior of travelling patterns in binary fluid convection [9]. Stationary solutions to the continuous one-dimensional cubic and cubic–quintic CGL equations can frequently be found by using both analytical and numerical methods. These equations can display interesting and complex dynamics including coherent structures [2]. One of the fundamental problems is to check these solutions against their stability, which is essential from a basic point of view as well as for potential applications [10]. In fact, the stability of such solutions has been studied for both the real [11,12] and complex [13,14] Ginzburg–Landau equations. Different kinds of instability may lead to such

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phenomena as bistability [15], self-oscillations, and the formation of static or moving patterns [16]. For example, linear stability analysis can determine the instability domain in parameter space and predict quantitatively how the amplitude of a modulation sideband evolves at the onset of the instability. However, such analysis is based on the linearization around the unperturbed carrier wave, which is valid only when the amplitude of perturbation is small in comparison with that of the carrier wave. Clearly, for long-time scales, when the instability is fully developed, the linear stability analysis fails and the modulated nonlinear plane waves can evolve into localized excitations. Hence, as is well known, a prominent instability is the modulation instability, which is the outcome of interplay between nonlinearity and dispersive effects. It is a symmetry-breaking instability so that a small perturbation on top of a constant amplitude background experiences exponential growth and this leads to wave breakup in either space or time. Since this disintegration typically occurs in the same parameter region where bright solitons are observed, modulational instability is considered to some extent, a precursor to soliton formation [17–21].

Besides the modulational instability, the Eckhaus instability which is responsible for partial wavelength selection in one-dimensional systems, can lead to spatio-temporal complexity [22]. Eckhaus [22] has shown that, for negative values of a control parameter ε, the only solution is the uniform or trivial state. At $\varepsilon=0$, the trivial solution loses stability to a periodic pattern of wavenumber k_c, that is of functional form $e^{ik_c x}$. At a slightly higher value of the control parameter, the trivial state is unstable to all periodic patterns e^{ikx} whose wavenumber satisfies $(k-k_c)^2 \le \varepsilon$. However, these periodic solutions are themselves unstable, unless k falls in the smaller range $(k-k_c)^2 \le \frac{1}{3}\varepsilon$, a parabolic region in the (k,ε) plane bounded on both sides by the unstable regions called “Eckhaus bands” [22]. The Eckhaus instability for travelling waves has been analysed in diverse convection problems, either experimentally [23–26], numerically or theoretically [24,27–31]. For example, two experimental papers have been devoted to the determination of the Eckhaus stability boundaries of travelling waves in binary fluid convection [23–26]. Other theoretical and numerical works have likewise considered the stability of extended pattern in binary fluid convection [30,31].

As is well known, spatial modulation leads for instance to a shift of the threshold [32]. In the special cases of the real Ginzburg–Landau equation with the sinusoidally modulated control parameter, it is found that the band of stable wave vectors is always reduced, with lower modulation frequencies giving greater reduction [32]. In general, temporal modulation may shift the threshold for the onset of the primary instability [33] can lead to pattern selection [34], and, in the presence of noise, can affect transitions between attractors [35]. Very recently, Bhadauria and Kiran [36] have shown that the dynamics of a weakly nonlinear oscillatory convection of viscoelastic fluid layer under gravity modulation is governed by a complex nonautonomous Ginzburg–Landau amplitude equation. In particular, it appears that modulation has a destabilization effect at low frequencies and a stabilization effect at high frequencies [36].

The main purpose of this article is to go beyond the spatially periodic contribution to the control parameter on a supercritical bifurcation from a homogeneous state to a spatially periodic state [27]. We focus on a subcritical bifurcation, where the cubic–quintic complex Ginzburg–Landau equation with real coefficients has been taken as a model equation with a modulated control parameter $\varepsilon \to \varepsilon + M(x)$. Then, by using the Poincaré–Lindstedt expansion, we show how, above the threshold, the quintic term has profound consequences on the stationary nonlinear solution as well as on the Nusselt number. In particular, the slope of the Nusselt number at the threshold decreases with increasing values of the square of the periodic spatial modulation amplitude, while the curvature of the Nusselt number also at the threshold increases with increasing values of the square of the periodic spatial modulation amplitude.

The paper is organized as follows. In Section 2, we introduce the model equation. The effects of the modulation on the threshold is calculated by using a perturbation method. The stationary nonlinear behavior of solutions is investigated by the Poincaré–Lindstedt expansion in Section 3. Next, the analytical predictions are compared with direct numerical simulations. Finally, Section 4 concludes the paper.

2. Model equation and the Poincaré–Lindstedt expansion theory

Let us write down the cubic–quintic CGL equation with real coefficients

$$\tau_0 \partial_t A = \left[\varepsilon + M(x) + \xi_0^2 \partial_x^2\right] A - g|A|^2 A - \gamma |A|^4 A. \tag{3}$$

This model is often employed in the description of patterns observed in Rayleigh–Bénard convection and pattern-forming systems in which Eq. (3) occurs naturally [2,21].

We assume that $\gamma>0$; $\xi_0^2>0$; $g>0$. The parameter $\gamma>0$ can be computed explicitly from the underlying system. $M(x)$ characterizes hydrodynamic systems with rough container boundaries or macroscopic porosity.

To a linear part of the real cubic–quintic Ginzburg–Landau equation, our results have common features with those reported by Hammele et al. [32]. For example, in the absence of modulations, the neutral curve, which provides an expression for the control parameter $\varepsilon(q)$ as a function of the wave number q, has a parabolic shape $\varepsilon_0(q)=\xi_0^2 q^2$, which takes its minimum at $q=0$, with the critical value $\varepsilon_c=\varepsilon_0(q=0)=0$. In the presence of modulations, the threshold is always negative, that is, $\varepsilon_c<0$, and lower than for the unmodulated system [32]. For the harmonic modulation $M(x)=2\eta\tilde{G}\cos(Qx)$, where η is a small parameter, which will be considered through this paper, the analytical formula for the threshold ε_c, is $\varepsilon_c^{(2)}=-2\tilde{G}^2/\xi_0^2 Q^2$ [32], where $G=\eta\tilde{G}$ is the amplitude modulation and Q is the wave number.

It is well known that there exists no general methodology to the integration of nonlinear ordinary differential equations, which, however, are the most important. Accordingly, numerous approaches for constructing approximate analytical solutions, most of which are perturbation techniques, have been developed [37]. These perturbation methods involve the expansion of a solution to a nonlinear ordinary differential equation in a power series in a so-called perturbation parameter. They include the Lindstedt–Poincaré (LP) method [37], the modified Lindstedt–Poincaré (LP) method [38], the Krylov–Bogoliubov–Mitropolsky (KBM) method [39], the multiple scales method [40] and the linearized harmonic balance method [41]. Recently, Yamgoue and Kofane have proposed a method consisting of a combination of the classical KBM method and of the modified LP method which provides high accurate result for undamped oscillations [42].

To describe the stationary nonlinear solutions exhibited by Eq. (3), we assume that the control parameter ε has large values than the amplitude of the modulation $M(x)$. In such condition of parametric modulations, the effects related to $M(x)$ become small and a perturbation expansion, which is valid immediately above the threshold, is possible. We introduce a small parameter λ that measures the distance from $\varepsilon_c(\eta)$ and the small amplitude of A close to threshold. We begin by assuming an expansion for the