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Solution of Riemann problem for dusty gas flow

R.K. Gupta*, Triloki Nath, L.P. Singh

Department of Mathematical Sciences, Indian Institute of Technology, Banaras Hindu University, Varanasi 221005, India

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ABSTRACT

A direct approach is used to solve the Riemann problem for a quasilinear hyperbolic system of equations governing the one dimensional unsteady planar flow of an isentropic, inviscid compressible fluid in the presence of dust particles. The elementary wave solutions of the Riemann problem, that is, shock waves, rarefaction waves and contact discontinuities are derived and their properties are discussed for a dusty gas. The generalised Riemann invariants are used to find the solution between rarefaction wave and the contact discontinuity and also inside rarefaction fan. Unlike the ordinary gasdynamic case, the solution inside the rarefaction waves in dusty gas cannot be obtained directly and explicitly; indeed, it requires an extra iteration procedure. Although the case of dusty gas is more complex than the ordinary gas dynamics case, all the parallel results for compressive waves remain identical. We also compare/contrast the nature of the solution in an ordinary gasdynamics and the dusty gas flow case.

1. Introduction

In gasdynamics, Riemann problem is an initial value problem for the system of one dimensional Euler equations supplemented by a discontinuous initial data. Its solution consists of three waves, with the middle wave as a contact discontinuity and the other two waves are shock or rarefaction waves depending upon the initial data. Also it gives us an idea of the wave structure of a system of hyperbolic partial differential equations. In recent decades solution of the Riemann problem for the Euler equations of ordinary gasdynamics has been analysed extensively. Lax [1] solved the Riemann problem by considering the difference between initial data, $\|V_L - V_R\|$, sufficiently small, where V_L and V_R are vectors of conserved variables at constant states separated by a discontinuity. Smoller [2] presented a solution of the Riemann problem for an extended class of hyperbolic systems with V_L and V_R to be arbitrary constant vectors. Glimm [3] used the solutions of Riemann problem in construction of a solution to the general initial value problem using the random choice method. Godunov [4] and Chorin [5] has proposed the exact solution of the Riemann problem, however, Smoller [6] has proposed a different approach to determine the exact solution. Liu [7] solved the Riemann problem for general system of conservation laws subject to entropy condition. Toro [8,9] presented a Riemann solver for the exact solution of the Riemann problem for ideal and covolume gases. Using the solution of the Riemann problem, Godunov [10] presented a numerical scheme for the solution of a nonlinear system of hyperbolic conservation laws. As the Riemann problem does not admit a solution in closed form, even for ideal gas, many authors, such as Godunov [4], Chorin [5], Smoller [6], Gottlieb and Groth [11], Quartapelle et al. [12] and Toro [13], among several others, developed iterative methods for the solution to determine the flow field. Menikoff and Plohr [14] studied the Riemann problem for fluid flow of real materials with arbitrary equation of state, subject to the physical requirements of thermodynamics like phase transition. Recently Shekhar and Sharma [15,16], Singh and Singh [17] presented the solution of Riemann problem for one dimensional magnetogasdynamics flow. Gupta and Singh [18] used random choice method for the solution of dam break problem which is an example of Riemann problem for shallow water equations. A detailed discussion on the Riemann problem can be found in the books Smoller [6], Toro [13], Li [19], Dafermos [20], Bressan [21] and LeVeque [22]. In the case of Euler equations Riemann problem contains the shock tube problem [23]. To determine the exact closed form solution to the Riemann problem for the Euler equations is still an open problem.

The study of Riemann problem for the fluid flow containing solid particles is a subject of great interest both from mathematical and physical point of view due to its applications such as in underground explosions [24], interstellar masses [25] and explosive volcanic eruptions [26] etc. Dusty gas is a mixture of gas and small solid particles where solid particles occupy less than 5% of total volume. When the speed of fluid is very high, the small solid particles behave like a pseudo fluid [27]. Miura and Glass [28] studied the flow resulting from the passage of a shock wave

* Corresponding author.
E-mail address: rkgupta.rs.apm12@itbhu.ac.in (R.K. Gupta).

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through a dusty-gas layer. The basics of gas particle flow can be found in [29]. The dynamical behaviour of a fluid is governed by the principle of conservation of mass, momentum and energy. Here we consider a single fluid model for dusty gas. The paper aims to provide an approximate analytical solution to the Riemann problem for the one-dimensional, time-dependent Euler equations for dusty gas flow. In case both external waves are rarefaction waves then it might create vacuum in the solution of Riemann problem depending upon the initial data. Using the Riemann solver of Toro [13] the non-vacuum solutions are determined, which is obtained if the pressure positivity condition is satisfied. It is also assessed as to how the presence of dust particles influences the solution across the shock wave, rarefaction wave and contact discontinuity.

2. Basic equations

The governing equations describing a planar flow of a dusty gas mixture obeying the equation of state of Mie Grüneisen type

$$p = (1 - k_p)\rho RT/(1 - Z), \tag{1}$$

are given as [27–30]

$$\begin{aligned} &\rho_t + v\rho_x + \rho v_x = 0,\\ &\rho(v_t + vv_x) + p_x = 0,\\ &p_t + vp_x + \rho c^2 v_x = 0, \end{aligned} \tag{2}$$

where v is the particle velocity along x-axis, t is the time, ρ is the density, p is the pressure, T is the temperature and R is the gas constant. The entity $Z = V_{sp}/V_g$ is the volume fraction and $k_p = m_{sp}/m_g$ is the mass fraction of the solid particles in the mixture where m_{sp} and V_{sp} are the total mass and volumetric extension of the solid particles and V_g and m_g are the total volume and total mass of the mixture respectively. The quantity

$$c = \left(\Gamma p/((1 - \theta\rho)\rho)\right)^{1/2}, \tag{3}$$

is the equilibrium speed of sound with

$$\Gamma = \gamma(1 + \lambda\beta)/(1 + \lambda\beta\gamma), \quad \lambda = k_p/(1 - k_p), \quad \beta = c_{sp}/c_p, \quad \gamma = c_p/c_v. \tag{4}$$

Here c_{sp} is the specific heat of the solid particles, c_p the specific heat of the gas at constant pressure, and c_v the specific heat of the gas at constant volume. The relation between the entities Z and k_p is given by $Z = \theta\rho$, $\theta = k_p/\rho_{sp}$, with ρ_{sp} as the species density of the solid particles.

The internal energy per unit mass of the mixture is given as

$$e = (1 - Z)p/((\Gamma - 1)\rho). \tag{5}$$

3. The Riemann problem and generalized Riemann invariants

The system of governing Eq. (2) along with (5) can be written in conservation form as

$$\left.\begin{aligned} &V_t + F(V)_x = 0,\\ &V = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix}, F(V) = \begin{bmatrix} \rho v \\ \rho v^2 + p \\ v(E + p) \end{bmatrix}, \end{aligned}\right\} \tag{6}$$

where $E = \rho e + \rho v^2/2$.

The initial conditions for the Riemann problem are

$$V(x, 0) = \begin{cases} V_L, & \text{if } x < 0 \\ V_R, & \text{if } x > 0 \end{cases}, \tag{7}$$

where $-\infty < x < \infty$, $t > 0$. We can take x to vary in a finite interval $[x_L, x_R]$ around the point $x = 0$. In the solution of Riemann problem, $U = (\rho, v, p)^T$ is taken as vector of primitive variables. The initial data of Riemann Problem (6)–(7) consists of two constant states, which are $U_L = (\rho_L, v_L, p_L)$ to the left of $x = 0$ and $U_R = (\rho_R, v_R, p_R)$ to the right of $x = 0$, separated by a discontinuity at $x = 0$. Physically, with reference to Euler equations, the shock-tube problem may be generalized as Riemann problem consisting of two stationary gases $(v_L = v_R = 0)$ in a tube separated by a diaphragm. When the diaphragm is broken down suddenly it produces a nearly centred wave system consisting of a rarefaction wave, a contact discontinuity and a shock wave.

The hyperbolic system of Eq. (6) admits the following family of characteristics:

$$dx/dt = v - c, \quad dx/dt = v, \quad dx/dt = v + c. \tag{8}$$

The family of characteristics given by second equation of (8) represents the particle path while those given by first and third represent the wave propagating in the negative and positive direction along x-axis, respectively. These three waves corresponding to Eq. (8) separate four constant states from left to right V_L, V_L^*, V_R^* and V_R. The unknown star region between the left and right waves is divided by the middle wave into two sub regions star left (V_L^*) and star right (V_R^*).

From the eigen structure of the Euler equations it can be easily seen that the middle wave is always a contact discontinuity while the left and right (nonlinear) waves are either rarefaction or shock waves. Thus according to the type of non-linear waves, there can be four possible wave patterns.

It can also be seen that both pressure p^* and particle velocity v^* are constant in the star region. Our solution procedure makes use of the constancy of pressure and velocity in the star region [13]. For the isentropic case, we can replace the third equation by the entropy equation

$$S_t + vS_x = 0.$$

Then the system (6), can be written as

$$\begin{bmatrix} \rho \\ v \\ S \end{bmatrix}_t + \begin{bmatrix} v & \rho & 0 \\ c^2/\rho & v & (1/\rho)\partial p/\partial S \\ 0 & 0 & v \end{bmatrix} \begin{bmatrix} \rho \\ v \\ S \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalues of the above system are

$\lambda_1 = v - c, \quad \lambda_2 = v, \quad \lambda_3 = v + c,$

and the corresponding right eigenvectors are

$$k^1 = \begin{bmatrix} 1 \\ -c/\rho \\ 0 \end{bmatrix}, \quad k^2 = \begin{bmatrix} -\partial p/\partial S \\ 0 \\ c^2 \end{bmatrix}, \quad k^3 = \begin{bmatrix} 1 \\ c/\rho \\ 0 \end{bmatrix}.$$

Across the wave associated with $\lambda_1 = v - c$, we have

$$\frac{d\rho}{1} = \frac{dv}{-c/\rho} = \frac{dS}{0},$$

which gives the relations

$$dv + (c/\rho)d\rho = 0 \text{ and } dS = 0.$$

i.e., $v + \int (c/\rho)d\rho =$ constant and $S =$ constant.

Similarly, across the $\lambda_3 = v + c$, wave we have

$v - \int (c/\rho)d\rho =$ constant and $S =$ constant, which are the generalized Riemann invariants for the system of Eq. (6).

4. Equation for pressure and velocity

To compute the pressure p, velocity v in the star region we