International Journal of Non-Linear Mechanics 82 (2016) 132–143

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Non-linear buckling of elliptical curved beams

Anh-Tuan Luu, Jaehong Lee*

Department of Architectural Engineering, Sejong University, 209 Neungdong-ro, Gwangjin-gu, Seoul 143-747, Republic of Korea

ARTICLE INFO

Article history:
Received 2 July 2015
Received in revised form
22 December 2015
Accepted 9 February 2016
Available online 17 February 2016

Keywords:
Non-linear buckling
Postbuckling
Elliptical curved beams
Isogeometric analysis

ABSTRACT

In this paper, a study on non-linear buckling and postbuckling behavior of elliptical curved beams is presented. An isogeometric analysis framework with the use of an arc-length iteration technique is developed based on the shear deformable theory and accounts for the geometric non-linearity in von Kármán sense. The proposed method is capable of exactly modeling the geometry of elliptical curved beams by means of NURBS (non-uniform rational B-splines) basis functions. The solutions of the present model are validated by comparing with experimental and numerical results of the literature. Numerical examples are carried out to investigate the effects of different geometric parameters on the buckling and postbuckling of elliptical curved beams with hinged–hinged, clamped–clamped and clamped–hinged boundary conditions.

1. Introduction

It is well known that the buckling behavior of curved beams is very complex since the beam deformations depend on the coupled equations between tangential and normal displacements and rotation by curvature effects. Up to the present, a large amount of work has been devoted to investigations of buckling response for curved beams. Many researchers including Timoshenko and Gere [1], Vlasov [2], Simitses [3], Papangelis and Trahair [5], Kang and Yoo [4], Kim et al. [6] and Attard et al. [7] attempted to obtain analytical solutions to the critical buckling load of circular curved beams under uniform radial compressive load. Similar problems were solved by Kang et al. [8] using differential quadrature method (DQM) and Yang et al. [9], Yoo and Kang [10], Kim et al. [11] and Öztürk et al. [12] by employing finite element method (FEM). In most of their works, the prebuckling that involves the flexural deformation was not considered. However, Pi and Trahair [13] showed that the effects of prebuckling deformation in the normal direction need to be taken into account for shallow circular curved beams under uniform compression since its relation to the beam rise is not small. Later, Pi et al. presented a series of studies [14–17] on non-linear buckling and postbuckling of circular curved beams carrying uniform radial compression with various geometries and boundary conditions. The non-linear stability of circular curved beams subjected to a central concentrated load was also analytically investigated by Bradford et al. [18], Pi et al. [19], and Pi and Bradford [20–22]. Zhu et al. [23] used the trapezoid method with Richardson extrapolation enhancement to trace non-linear equilibrium paths for circular curved beams with different loading cases. Chandra et al. [24] presented a combined experimental–computational framework to analyze the non-linear transient response of clamped shallow circular curved beams and determine the snap-through boundaries in the parameter space. It should be noted that in the above-mentionedstudies, the radius of curvature of curved beams is constant, thus the problems can be simplified by ignoring the variable curvature effect in integrations. For non-circular curved beams, Dinnik [25] provided one of the earliest numerical solutions to the linear buckling of parabolic curved beams under a uniformly distributed vertical load. The similar problems were treated by Tadjbakhsh [26], Attard et al. [27] and Luu et al. [28]. While Tadjbakhsh [26] used the Euler–Bernoulli assumptions in his research, Attard et al. [27] and Luu et al. [28] improved their formulations by taking into account the shear deformation effect. Moon et al. [29], Bradford et al. [30], Cai and Feng [31], Cai et al. [31,32] and Zhu et al. [33] considered geometric non-linearities in their buckling analyses of parabolic curved beams under the same loading case with the previously mentioned studies [25–28]. Chandra et al. [34] investigated dynamic transitions associated with the snap-through buckling for sinusoidal curved beams subjected to a sinusoidal distributed load. Nieh et al. [35] constructed an analytical solution to the buckling load of elliptical curved beams by incorporating a series of solutions and stiffness matrices when they are applied by a uniformly distributed vertical load. Both the effects of geometric non-linearity and shear deformation were neglected in the study of Niel et al. [35].

Recently, a family of the so-called isogeometric analysis firstly introduced by Hughes et al. [36] has been seen as a powerful

* Corresponding author.
E-mail addresses: tuanla.hn@gmail.com (A.-T. Luu), jhlee@sejong.ac.kr (J. Lee).

http://dx.doi.org/10.1016/j.ijnonlinmec.2016.02.001
0020-7462/

computational method. The main goal of isogeometric analysis (IGA) is to offer a possibility to fulfill the seamless link between computer aided design (CAD) and finite element analysis (FEA). The underlying concept of IGA is to utilize basis functions that are popular in modern CAD software for numerical simulations of physical phenomena. Among others, non-uniform rational B-splines have been mostly employed in IGA due to their popularity and their capability to exactly represent the conic sections. In a recent book, Cottrell et al. [37] explained in detail the isogeometric concept and presented systematic procedures to apply IGA in structural analysis and fluid mechanics. Isogeometric finite elements were proven to have superior computational performances than the conventional finite elements [38–40]. Isogeometric analysis has been used to efficiently solve many problems in a wide range of research areas [28,41–48]. In structural buckling analysis, conventional finite elements are notoriously sensitive to mesh distortion and geometric imperfection. The ability of exact geometry simulation with the mesh smoothness lets IGA become an ideal candidate in solving buckling problems [28,43,49–55].

From the previously cited references, one can note that despite extensive research for the buckling analysis of circular, parabolic and sinusoidal curved beams, to the authors' knowledge, Ref. [35] is the only study reported on the buckling of elliptical curved beams with lack of a geometric non-linearity in the open literature. Therefore, our objective is to investigate the buckling and postbuckling behavior of elliptical curved beams using a NURBS-based isogeometric analysis framework. The elliptical curved beams are subjected by a central concentrate vertical load and all clamped–clamped, hinged–hinged and clamped–hinged boundary conditions. Governing equations are derived in the framework of shear deformable curved beam theory and the geometric non-linearity is accounted for in the von Kármán sense. The geometry of elliptical curved beams is exactly represented by NURBS and NURBS basis functions are used in approximating the unknowns. An arc-length iteration method incorporated with the predictor criterion proposed by Feng et al. [56,57] is implemented to solve non-linear equilibrium equations. The present study is validated by comparing the current results with experimental and numerical results previously published in the literature for a circular curved beam and an elliptical curved beam. The influences of geometrical parameter, slenderness ratio and rise-to-span ratio on the buckling characteristics of elliptical curved beams are investigated in detail through extensive parametric studies. It is shown that all bifurcation, snap-through and snap-back buckling can occur for elliptical curved beams.

2. Non-linear buckling isogeometric analysis of elliptical curved beams

In this section, a geometrically non-linear isogeometric formulation is developed for investigating the buckling and post-buckling behavior of elliptical curved beams. The exact geometry of elliptical curved beams is represented by NURBS. An arc-length iteration method is employed to trace possible bifurcation, snap-through and snap-back buckling phenomena. First, the main features of NURBS curve and NURBS basis functions are overviewed. The details for NURBS-based geometric modeling could be found in Piegl and Tiller [58] and Cottrell et al. [37].

2.1. A brief review of NURBS

NURBS are derived from B-splines which are piecewise polynomial curves composed of linear combinations of B-spline basis functions. The primary component of B-spline basis functions is a knot vector $\Xi = \left[\xi_1, \xi_2, \ldots, \xi_{n+p+1}\right]$ which is a set of non-decreasing real numbers in the interval $[\xi_1, \xi_{n+p+1}]$. Here, $\xi_i \in \mathbb{R}$ is the ith knot, i is the knot index, $i = 1, 2, \ldots, n+p+1$, p is the polynomial order and n is the number of basis functions. If all knots are equally spaced, the knot vector is called uniform. Otherwise, it is a non-uniform knot vector. If the first and last knots are repeated $(p+1)$ times, it is said to be open. The multiplicity of a knot value is the number of times it appears in the knot vector. The intervals $\left[\xi_1, \xi_{n+p+1}\right]$ and $[\xi_i, \xi_{i+1})$ are called a patch and a knot span, respectively. Based on the knot vector Ξ, B-spline basis functions $B_{i,p}(\xi)$ are defined recursively starting with piecewise constants $(p=0)$ as follows:

$$B_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

For $p = 1, 2, 3, \ldots$, they are defined by

$$B_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} B_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1,p-1}(\xi). \tag{2}$$

At a knot of multiplicity k, the B-spline basis functions are $C^{p-k}-$continuous at that knot. In addition, the basis functions are interpolatory at the ends of the interval and at the knot whose multiplicity is p (C^0-continuous). When p=1, the B-spline basis functions are identical to the standard Lagrange linear finite element basis functions.

A piecewise polynomial B-spline curve $\mathbf{C}(\xi)$ is defined by a linear combination of B-spline basis functions and coefficients over the parametric space:

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} B_{i,p}(\xi) \mathbf{P}_i. \tag{3}$$

The coefficients are points in d-dimensional physical space $\mathbb{R}^d$, referred to as control points. The control points $\mathbf{P}_i \in \mathbb{R}^d$, $i = 1, 2, \ldots, n$, construct the control polygon. An example of a B-spline curve constructed by eight B-spline basis functions of the open, non-uniform knot vector $\Xi = \{0, 0, 0, 1, 2, 3, 3, 4, 5, 5, 5\}$ is given in Fig. 1.

A rational B-spline curve is defined as follows:

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} \frac{B_{i,p}(\xi) w_i \mathbf{P}_i}{\sum_{i=1}^{n} B_{i,p}(\xi) w_i} = \sum_{i=1}^{n} N_{i,p}(\xi) \mathbf{P}_i, \tag{4}$$

where w_i and $N_{i,p}$ are weights and rational B-spline basis functions, respectively. If the knot vector is non-uniform, the basis becomes non-uniform rational B-spline basis functions.

2.2. Weak formulation

An elliptical curved beam governed by the Timoshenko theory is considered in Fig. 2. The geometry of the curved beam is illustrated in a right-handed orthogonal curvilinear coordinate system (s, r). The s-curvilinear coordinate coincides with the centroidal axis of the curved beam and κ denotes the curvature (R is the radius of curvature); u and w are the tangential and normal (radial) displacements, respectively, at the centroidal axis; θ is the rotation of the beam cross-section about the out-of-plane axis. Considering the extensional, flexural and shear deformations, the von Kármán strain–displacement relations are given by:

$$\epsilon = u' + \kappa w + \tfrac{1}{2}(w' - \kappa u)^2 \tag{5a}$$

$$\chi = \theta' \tag{5b}$$

$$\gamma = \theta + w' - \kappa u \tag{5c}$$

where ϵ and χ are the membrane strain and the curvature strain, respectively; γ is the transverse shear strain; the superscript “prime” refers to differentiation with respect to s.