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Linear hydraulic fracture with tortuosity: Conservation laws and fluid extraction

M.R.R. Kgatle *, D.P. Mason

School of Computer Science and Applied Mathematics, DST-NRF Centre of Excellence in Mathematical and Statistical Sciences, University of the Witwatersrand, Johannesburg, Private Bag 3, Wits 2050, South Africa

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ABSTRACT

Fluid extraction from a pre-existing two-dimensional hydraulic fracture with tortuosity is investigated. The tortuous fracture is replaced by a symmetric open fracture without asperities (deformations) on opposite crack walls but with a modified Reynolds flow law and a modified crack law (the linear crack law). The Perkins–Kern–Nordgren approximation is made in which the normal stress at the fracture walls is proportional to the half-width of the symmetric model fracture. By using the multiplier method two conservation laws for the non-linear diffusion equation for the half-width are derived. Two analytical solutions generated by the Lie point symmetries associated with the conserved vectors are obtained. One is the known solution for a fracture with constant volume. The other is new and is the limiting solution for fluid extraction. A jet of fluid escapes from the fracture entry and the volume of the fracture decreases. There is a dividing cross-section between fluid flowing towards the fracture entry and fluid flowing towards the fracture tip which explains why the length of the fracture continues to grow as fluid is extracted. As tortuosity increases the position of the dividing cross-section moves closer to the entry. A numerical solution is presented for the other cases of fluid extraction. Comparison of the fluid flux for different operating conditions within the fluid extraction region shows that the limiting solution yields the maximum rate of fluid extraction from the fracture. As the fracture becomes more tortuous its length becomes less dependent on the operating conditions at the fracture entry. For fluid extraction working conditions close to the constant volume operating condition the width averaged fluid velocity increases approximately linearly along the whole length of the fracture. For these working conditions, an approximate analytical solution for the half-width for fluid extraction, which agrees well with the numerical solution, is derived by assuming that the width averaged fluid velocity increases exactly linearly along the fracture.

1. Introduction

In an earlier paper which we refer to as paper [1], we considered the propagation of a pre-existing fluid-filled linear hydraulic fracture with tortuosity. The propagation of the fracture for fluid injection into the fracture at the entry was investigated. In this paper, we consider further the linear hydraulic fracture with tortuosity. We first investigate conservation laws for the non-linear diffusion equation for the fracture half-width. As well as the elementary conservation law, a second conservation law for the partial differential equation is found. Analysis of the physical significance of the second conservation law leads to investigation of a tortuous hydraulic fracture with fluid extraction at the entry.

Tortuosity occurs in a fracture when fluid flows in intricate paths due to complicated fracture geometries resulting from the presence of asperities at the fluid rock interface. The modified Reynolds' flow law [2] was used to model the tortuosity in the fluid flow. Contact regions are formed when the fluid pressure is insufficient to support the normal stresses and asperities on the upper and lower surfaces of the fracture touch each other. The linear crack law models these contact regions. The normal stresses are thus supported by both the fluid pressure and the areas of contact.

In Section 2, a summary of the modelling process for a tortuous hydraulic fracture, which is presented fully in paper [1], is given and the equations required in this paper are stated. Section 3 focuses on the conservation laws for the non-linear diffusion equation for the fracture half-width. The derivation of the conservation laws using the multiplier method is outlined. Lie point symmetries are associated with the corresponding conserved vectors. In Section 4 the analytical solutions generated by the

* Corresponding author. Tel.: +27 11 717 6111.
E-mail address: mrr.kgatle@gmail.com (M.R.R. Kgatle).

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associated Lie point symmetries are derived. One is the known solution for a hydraulic fracture evolving with constant volume. The other is a new solution for fluid extraction. In Section 5 the physical properties of the new solution for fluid extraction are investigated. The volume flux along the fracture and the total volume of the fracture are considered. In Section 6 the numerical solution for fluid extraction is presented. For operating conditions between the limiting value for fluid extraction and the value for constant volume the boundary value problem has to be solved numerically. In Section 7 the width averaged fluid velocity is considered. It varies approximately linearly along the length of the fracture for part of the range of extraction operating conditions. An approximate analytical solution is derived based on the assumption that the width averaged fluid velocity varies exactly linearly along the fracture. Finally the conclusions are summarised in Section 8.

2. Model formulation

In this section, we will briefly review the modelling process for a linear hydraulic fracture with tortuosity and summarise the results of paper [1] which will be required in this paper.

The tortuous fracture with asperities at the fluid–rock interface is replaced by a symmetric two-dimensional fracture illustrated in Fig. 1 but with a modified Reynolds flow law to account for the effect of asperities at the fluid–rock interface on the fluid flow. The lubrication approximation for a long thin fracture was made. In order to close the system of equations we used both the linear crack law and the Perkins–Kern–Nordgren (PKN) approximation [3,4] in which a linear relation between the fluid pressure $p(t,x)$ and the half-width of the model symmetric fracture $h(t,x)$ is assumed. An open fracture and a partially open fracture are defined in terms of the maximum and minimum half-widths, h_{max} and h_{min}. For an open fracture with no contact areas formed by touching asperities, $h(t,x) \geq h_{max}$. For a partially open fracture $h_{min} \leq h(t,x) < h_{max}$. We consider fractures for which $h_{min} \ll h_{max}$ and we make the approximation that $h_{min} = 0$. For an open fracture the normal stress at the crack wall, $\sigma_{zz}(t,x)$, is supported entirely by the fluid pressure and therefore the effective stress, $\sigma_{zz}(t,x)+p(t,x)$, vanishes. For a partially open fracture, the linear crack law models the contact regions due to the touching asperities. In the linear crack law $h(t,x)$ is related to the effective stress, $\sigma_{zz}(t,x)+p(t,x)$, by a piecewise linear law. The volume flux of fluid across the fracture per unit breadth, $Q(t,x)$, satisfies the modified

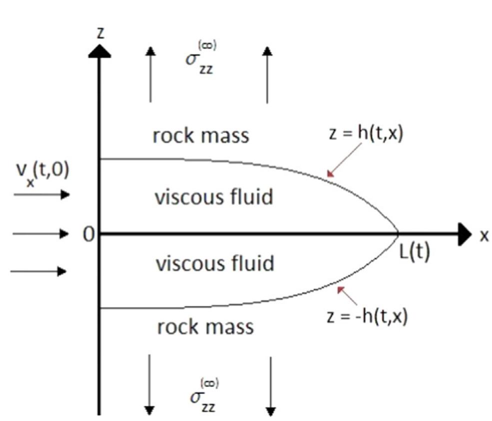

Fig. 1. Hydraulic fracture without asperities on the crack walls or contact regions, satisfying the Reynolds flow law.

Reynolds flow law

$$Q(t,x) = -\frac{2}{3\mu} a_n h^n \frac{\partial p}{\partial x}, \tag{2.1}$$

where μ is the coefficient of viscosity of the fluid and n and a_n are constants determined from experiment. For Reynolds flow, $n=3$ and $a_3 = 1$.

The half-width h_{max} is chosen as the characteristic half-width of the fracture and is used to make the half-width of the model symmetric fracture $h(t,x)$ dimensionless. The half-width $h(t,x)$ satisfies the dimensionless non-linear diffusion equation:

$$\frac{\partial h}{\partial t} = K_n \frac{\partial}{\partial x}\left(h^n \frac{\partial h}{\partial x}\right), \quad K_n = \frac{a_n h_{max}^{n-3}}{3}\left(\frac{1-\dfrac{\sigma_R}{\Lambda h_{max}}}{1-\dfrac{\sigma_{zz}^{(\infty)}}{\Lambda h_{max}}}\right), \tag{2.2}$$

where σ_R is the least effective stress for which $h(t,x) = 0$ and $\sigma_{zz}^{(\infty)}$ is the normal stress at $z = \pm\infty$. For an open fracture $\sigma_R = 0$ while for a partially open fracture $\sigma_R < 0$. The coordinate system is such that the x-coordinate is along the length of the fracture, the z-axis is along the width of the fracture and the y-axis is along the breadth of the fracture (Fig. 1). All quantities are independent of y.

The Lie point symmetry generator of the partial differential equation (2.2) is:

$$X = (c_1 + c_2 t)\frac{\partial}{\partial t} + (c_3 + c_4 x)\frac{\partial}{\partial x} + \frac{1}{n}(2c_4 - c_2)h\frac{\partial}{\partial h}, \tag{2.3}$$

where c_1, c_2, c_3 and c_4 are constants. We choose $c_3 = 0$ so that the similarity variable $u=0$ when $x=0$. The invariant boundary value problem is:

$$\frac{d}{du}\left(f^n \frac{df}{du}\right) + \frac{d}{du}(uf) + \frac{1}{n}\left(\frac{1}{\alpha} - (n+2)\right)f = 0, \tag{2.4}$$

$$f(1) = 0, \tag{2.5}$$

$$f^n(0)\frac{df}{du}(0) = \frac{1}{n}\left(\frac{1}{\alpha} - (n+2)\right)\int_0^1 f(u)\, du, \tag{2.6}$$

where

$$\alpha = \frac{c_4}{c_2}, \quad u = \frac{x}{L(t)} \tag{2.7}$$

and $L(t)$ is the length of the hydraulic fracture. The constant α is determined by the working conditions at the fracture entry $x=0$. The initial length of the fracture is non-zero and is the characteristic length of the fracture. The length obtained from the boundary condition at the fracture tip is

$$L(t) = \left(1 + \frac{c_2}{c_1} t\right)^{\alpha}. \tag{2.8}$$

Once $f(u)$ has been found by solving the boundary value problem (2.4)–(2.6), the volume of the fracture per unit breadth is given by

$$V(t) = V_0 \left(1 + \frac{c_2}{c_1} t\right)^{((n+2)/n)(\alpha - 1/(n+2))}, \tag{2.9}$$

where

$$V_0 = 2\left(\frac{c_4}{K_n c_1}\right)^{1/n} \int_0^1 f(u)\, du. \tag{2.10}$$

The half-width of the fracture is

$$h(t,u) = \left(\frac{c_4}{K_n c_1}\right)^{1/n} \left(1 + \frac{c_2}{c_1} t\right)^{(2/n)(\alpha - 1/2)} f(u). \tag{2.11}$$

The volume flux per unit breadth is

$$Q(t,u) = \tag{2.12}$$