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Simple shearing and azimuthal shearing of an internally balanced compressible elastic material

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ABSTRACT

The finite deformation response of a compressible internally balanced elastic material is studied for deformations that involve progressive shearing. The internally balanced material theory requires that an equation of internal balance is satisfied at each material point. This arises from the constitutive theory which makes use of a multiplicative decomposition of the deformation gradient. Satisfaction of the internal balance requirement then yields the most energetically favorable decomposition. Here we consider a particular compressible internally balanced material model that is motivated by a Blatz–Ko type energy from the conventional hyperelastic theory. The conventional hyperelastic theory occurs as a special limiting case of the internally balanced constitutive theory. More generally, the internally balanced material exhibits softer mechanical behavior. This gives rise to a stress-plateau in the simple shearing response whereas such plateaus do not occur in the corresponding hyperelastic treatment. The boundary value problem for azimuthal shearing with a possible radial stretching is then studied. The internally balanced material response is again found to be softer than that of the hyperelastic limiting case. This is manifest in terms of an upper bound to the applied twisting moment for the existence of solutions to the boundary value problem. In contrast, the hyperelastic limiting case has solutions for all values of applied moment.

1. Introduction

In finite deformation solid mechanics the deformation gradient $\mathbf{F}=\partial\mathbf{x}/\partial\mathbf{X}$ is central to the kinematic description. Here $\mathbf{x}=\chi(\mathbf{X})$ is the mapping from the reference location $\mathbf{X}$ to the current location $\mathbf{x}$. The theory of hyperelasticity makes use of $\mathbf{F}$ to develop its constitutive theory in terms of the elastic stored energy density $W=W(\mathbf{C})$ where $\mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{F}$. In the absence of internal material constraints, the Cauchy stress $\mathbf{T}$ is then given by

$$\mathbf{T}=\frac{2}{J}\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} \tag{1}$$

with $J=\det\mathbf{F}$. The stress equations of equilibrium in the absence of body forces take the well known form

$$\operatorname{div}\mathbf{T}=\mathbf{0} \tag{2}$$

where div is the divergence operator with respect to current configuration $\mathbf{x}$.

More general treatments of solid material behavior, specifically those that seek to describe how a combination of elastic and inelastic effects govern large deformation, often make use of a multiplicative decomposition of $\mathbf{F}$, say

$$\mathbf{F}=\hat{\mathbf{F}}\mathbf{F}^{*}. \tag{3}$$

This includes the Kröner–Lee multiplicative decomposition for the treatment of finite deformation plasticity [1,2], as well as descriptions of growth and remodeling in biological tissue (e.g., [3,4]). The standard modeling scenario when invoking (3) involves $\mathbf{F}^{*}$ describing the inelastic part of the process after which $\hat{\mathbf{F}}$ provides some elastic accommodation. The scientific literature in this area is now vast, and new types of physical phenomena are regularly being described using such a decomposition [5–7]. Because elastic and inelastic effects may permeate all aspects of a complex physical process, decomposition sequences in which elastic and inelastic factors alternate with each other can also logically be considered (e.g., [8]). This motivates the consideration of (3) in a context where both $\hat{\mathbf{F}}$ and $\mathbf{F}^{*}$ are each associated with a separate purely elastic type of effect. A theory of internally balanced elastic materials emerges under such considerations. Because the conventional theory of hyperelasticity (meaning the theory which does not invoke (3)) provides useful simplifications under the

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constraint of material incompressibility, a theory of internally balanced incompressible elastic solids was first presented in [9,10]. These works showed how the essential features of the decomposition (3) are determined in an equilibrium setting by minimization with respect to the decomposition (3) itself.

More recently, we have established how a theory of internally balanced elastic materials based on (3) emerges when the solid is subject to no material constraints whatsoever [11]. Here it is useful to recall the situation in conventional isotropic hyperelasticity, where the incompressible material theory is typically mathematically more tractable than the compressible material theory. The presence of the Lagrange multiplier pressure in the incompressible material theory often allows easy mathematical eliminations that simplify both the determination of basic force–stretch relations (in the context of homogeneous deformations) and the getting of simple integrable governing equations (in the context of boundary value problems describing inhomogeneous deformation). Indeed the incompressible theory leads to broad classes of universal inhomogeneous deformations that satisfy (2) whereas no such universal inhomogeneous deformations exist in the compressible hyperelastic material theory.

In the internally balanced elastic material theory for which (3) is an essential feature, comparing the compressible theory development of [11] to the incompressible theory development of [9,10] shows how the compressible theory generally requires more involved analytical procedures because of the extra degree of freedom associated with an unspecified volume change. Even so, it was shown in [11] how the compressible material theory still leads to tractable analysis of key homogeneous deformation behavior. The work [11] also showed how the conventional hyperelastic theory is naturally retrievable in a well defined limiting sense of the more general internally balanced material theory. It was further established how the internally balanced material theory provides for an additional softening mechanism that manifests itself for large strains.

While treating the important cases of uniaxial loading and pure pressure loading, the development of [11] did not treat deformations for which the principle directions change as the deformation proceeds. This is the typical situation, for example it occurs even in a simple shearing. Nor did [11] consider states of inhomogeneous deformations as would naturally arise in the consideration of boundary value problems. We address these issues here by focusing on shearing deformations. A strong motivation for our treatment is provided by the work of Wineman and Waldron [12] who show, in the context of conventional hyperelasticity, how a thorough knowledge of the normal stress effects in homogeneous shearing deformations can guide the analysis of boundary value problems in which shearing arises due to various twisting actions that occur on external boundaries. We find that a similar understanding can guide the analysis for internally balanced compressible elastic solids. Specifically, we show how the simple shearing mechanical response (shear stress vs. amount of shear) gives a large shear response in which the shear stress approaches an asymptotic upper bound. The asymptotic bound is dependent upon material parameters in the constitutive law such that special limiting choices of the material parameters recover a conventional hyperelastic theory in which the shear stress increases without bound. Thus the general softening aspect for this constitutive class of internally balanced elastic solids is confirmed for shearing deformations. These aspects are described in Sections 2–4.

Guided by these results a boundary value problem for the twisting of a cylinder is then formulated so as to yield deformations in which azimuthal shearing is accompanied by radial deformation. The boundary value problem is treated and solved by a numerical shooting algorithm in Sections 5–6. It is found that the radial part of the deformation involves volume decrease near the inner boundary and volume increase near the outer boundary. Such solutions are only obtainable for a finite range of applied twisting amount. The moment–twist relation correlates with the simple shearing behavior in the sense that materials with larger shear stress asymptotes in simple shear are found to generate a larger range of twisting moments with equilibrium solutions. Broader connections are described in the concluding Section 7.

2. Background

The elastic stored energy density W for an internally balanced material depends upon both $\hat{\mathbf{F}}$ and $\mathbf{F}^*$ from (3). This is equivalent to a dependence upon $\mathbf{F}$ and $\mathbf{F}^*$. As discussed in [9] the requirement of material frame indifference then gives that $W = W(\mathbf{C}, \mathbf{C}^*)$ where $\mathbf{C}^* = \mathbf{F}^{*\mathrm{T}}\mathbf{F}^*$. It is also useful to point out that a dependence on $\mathbf{C}$ and $\mathbf{C}^*$ is not equivalent to a dependence on $\mathbf{C}$ and $\hat{\mathbf{C}} = \hat{\mathbf{F}}^{\mathrm{T}}\hat{\mathbf{F}}$. Integrating W over the whole body gives the overall energy in the system. Equilibrium configurations minimize this energy with respect to both $\mathbf{x}$ and $\mathbf{C}^*$.

Eqs. (1) and (2) continue to hold in the equilibrium theory of internally balanced elastic materials because they follow from the minimization with respect to $\mathbf{x}$. Energy minimization with respect to $\mathbf{C}^*$ generates the internal balance relation

$$\frac{\partial W}{\partial \mathbf{C}^*} = 0, \tag{4}$$

which is the feature that distinguishes the internally balanced material theory from conventional hyperelasticity.

Recall also that if a hyperelastic material is isotropic then the dependence of W on $\mathbf{C}$ is only through the principal scalar invariants I_1, I_2 and $I_3 = J^2$ where

$$I_1 = \mathrm{tr}(\mathbf{C}), \quad I_2 = \tfrac{1}{2}\left[(\mathrm{tr}(\mathbf{C}))^2 - \mathrm{tr}(\mathbf{C}^2)\right], \quad I_3 = \det \mathbf{C}, \tag{5}$$

so that $W = W(I_1, I_2, I_3)$. In this case (1) gives

$$\mathbf{T} = \frac{2}{J}\left[\frac{\partial W}{\partial I_1}\mathbf{B} + \frac{\partial W}{\partial I_2}\left(I_1\,\mathbf{B} - \mathbf{B}^2\right) + I_3\,\frac{\partial W}{\partial I_3}\,\mathbf{I}\right] \tag{6}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{I}$ is the identity tensor.

As discussed in [11], a sufficient condition for an internally balanced elastic material to be isotropic is that the dependence of W on $\mathbf{C}$ and $\mathbf{C}^*$ is only through I_1^*, I_2^*, I_3^*, which are the principal scalar invariants of $\mathbf{C}^*$, and $\hat{I}_1$, $\hat{I}_2$, $\hat{I}_3$, which are the principal scalar invariants of $\hat{\mathbf{C}}$. With respect to the latter it is important to note that $\hat{I}_1, \hat{I}_2$ and $\hat{I}_3$ can each be determined directly from $\mathbf{C}$ and $\mathbf{C}^*$ even though $\hat{\mathbf{C}}$ itself cannot. For example

$$\hat{I}_1 = \mathrm{tr}\,\hat{\mathbf{C}} = \hat{\mathbf{F}} : \hat{\mathbf{F}} = (\mathbf{F}\mathbf{F}^{*-1}) : (\mathbf{F}\mathbf{F}^{*-1}) = \mathbf{F}^{*-1}\mathbf{F}^{*-\mathrm{T}} : \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{C}^{*-1} : \mathbf{C}. \tag{7}$$

Taking $W = W(\hat{I}_1, \hat{I}_2, \hat{I}_3, I_1^*, I_2^*, I_3^*)$ the derivatives in (1) and (4) are then calculated from the chain rule

$$\frac{\partial W}{\partial \mathbf{C}} = \sum_{i=1}^{3} \frac{\partial W}{\partial \hat{I}_i}\frac{\partial \hat{I}_i}{\partial \mathbf{C}} \quad \text{and} \quad \frac{\partial W}{\partial \mathbf{C}^*} = \sum_{i=1}^{3} \frac{\partial W}{\partial \hat{I}_i}\frac{\partial \hat{I}_i}{\partial \mathbf{C}^*} + \sum_{i=1}^{3} \frac{\partial W}{\partial I_i^*}\frac{\partial I_i^*}{\partial \mathbf{C}^*}. \tag{8}$$

On this basis it is found [11] that (1) gives

$$\mathbf{T} = \frac{2}{J}\left[\frac{\partial W}{\partial \hat{I}_1}\hat{\mathbf{B}} + \frac{\partial W}{\partial \hat{I}_2}\left(\hat{I}_1\hat{\mathbf{B}} - \hat{\mathbf{B}}^2\right) + \hat{I}_3\,\frac{\partial W}{\partial \hat{I}_3}\,\mathbf{I}\right] \tag{9}$$

with $\hat{\mathbf{B}} = \hat{\mathbf{F}}\hat{\mathbf{F}}^{\mathrm{T}}$. The internal balance equation (4) then takes the form

$$\boldsymbol{\Xi} = \mathbf{0}, \tag{10}$$