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On a mechanical lens

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ABSTRACT

In this paper, we consider the dynamics of a heavy homogeneous ball moving under the influence of dry friction on a fixed horizontal plane. We assume the ball to slide without rolling. We demonstrate that the plane may be divided into two regions, each characterized by a distinct coefficient of friction, so that balls with equal initial linear and angular velocity will converge upon the same point from different initial locations along a certain segment. We construct the boundary between the two regions explicitly and discuss possible applications to real physical systems.

1. Introduction

Systems with friction continue to be an area of intense interest. Modern computer methods and high-accuracy measuring equipment offer new possibilities for the analysis of systems with friction. A detailed analytical and experimental investigation of the motion of body with a flat axisymmetric base on a horizontal plane in the presence of dry friction forces is presented in [1,2]. It is shown that the real motion of this system does not fit in with any of the well-known models of pressure distribution of a body on a plane which are used in theoretical studies. In the same paper a systematic review of the well-known results in this area can be found. The motion and controllability of a ball in the presence of friction are examined in [3] by inserting rotors and in [4] by inserting internal moving masses. New problems relating to systems with friction also include the dynamics of mobile robots [5,6], manipulators [7], and vibro-impact dynamics [8,9], granular materials [10,11], etc. It is well-known that friction is of fundamental importance in problems involving sports dynamics such as billiards [12], bowling [13,14], curling [15,16], motion of the skateboard [17], etc.

The effects of friction are sometimes described using the non-holonomic model. This model is a simplification of the initial systems with friction: the coefficient of friction tends to infinity and the motion is assumed to occur on an absolutely rough plane. The classical results on non-holonomic dynamics which go back to the work of Routh, Appell, Chaplygin, Zhukovskii, and others (see e.g. [18,19]) are well-known. Some recent results on the dynamics of non-holonomic systems can be found in the paper by Batista [20,21], where the motion of disks on a plane is studied, and in the paper [22], where the motion of a ball is considered. It should be noted that the behavior of such non-holonomic systems exhibits strange, unusual dynamics. The problem with especially demonstrative behavior in this sense is the motion of Celtic stone [23]. Furthermore, the presentation of non-holonomic constraint as very large coefficient of friction may be incorrect: as was shown by Painleve [24], the dynamics may depend essentially on the value of this coefficient and even become contradictory.

It is well-known that sliding and rolling phases can alternate in the problem of motion of a ball on a plane. The most famous and popular dynamical game based on the dynamical properties of the ball is bowling.

It seems that a ball thrown from the hands of a professional works miracles. However, theoretical and applied studies carried out over the last fifty years have shown that, on the one hand, under the simplest initial conditions and parameters bowling is quite a determined game, but, on the other hand, there are a lot of unexplained effects observed in professional bowling which still remain unexplained. The motion of a ball in professional bowling can be divided into two phases. The first one is a slightly curled (the ball moves in a parabola) sliding of the ball on a fairly smooth oiled surface with coefficient of friction about ~ 0.04. The second one is the passage of the ball to a dry surface (the coefficient of friction is ~ 0.2) with subsequent rolling without slipping and a remarkable phenomenon of hook – a sharp curl of the trajectory during the terminal motion (Fig. 1).

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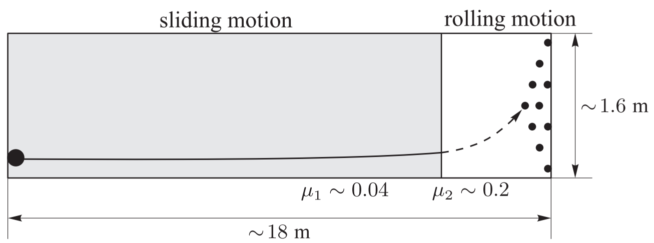

Fig. 1. Scheme of bowling. Oiled path of sliding (the solid line) and rough path of rolling (the dashed line) of a ball.

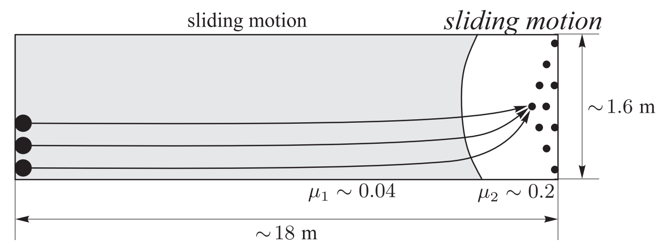

Fig. 2. Scheme of a "mechanical lens" in bowling. The balls slide from the starting points to the target point without rolling.

While the first stage is a well-known effect of motion of a ball in a parabola during sliding, studied by Euler [25], the second stage is a more complex motion studied in a large body of literature. It is well-known that a homogeneous ball rolls in a straight line [18], this fact is also illustrated with bowling [26]. But this effect can be observed in unprofessional amateur bowling, where paths and balls are not prepared in a special way. A great deal of classical research is devoted to the dynamics of a ball with non-uniform mass distribution [19,22], where it is shown that the trajectory of a ball deviates from a straight line during the rolling motion. Some papers are directly concerned with the explanation and prediction of the dynamics of special professional bowling balls [13,14] with emphasis on the final instant of the ball's motion – hook.

However, as stated above, the ball starts a curl at the stage of sliding. This curling depends on the coefficient of friction. The question arises whether we can reach the effect of hook on the stage of sliding before the transition to rolling, for example, when the ball passes from the smoother to the rougher surface of the path. It is also of interest to consider a more complicated problem of a sliding ball. Assuming the coefficient of friction to be variable, we could calculate the boundary between the surfaces in the path in such a way that parallel families of trajectories of sliding analogous balls with equal initial conditions (linear and angular velocities) converges to a predetermined point, for example, to the central skittle in the bowling[1] (Fig. 2). We call this phenomenon – the effect of "mechanical lens" by analogy with the optical effect of the well-known collecting lens focusing the light beams in one point.

Thus, this paper is devoted to analytical and numerical studies of the effect of the curling of a trajectory and the effect of a "mechanical lens" in the dynamics of a ball during sliding. Also, we study a possible application of these phenomena to the bowling game.

2. Equations of motion of a sliding ball

Consider a heavy homogeneous ball moving by inertia on a fixed rough horizontal plane. We assume the velocity of the point of contact to be sufficiently large to neglect the spinning friction and rolling friction and their influence on the law of sliding friction. For the latter we take the Coulomb law

$$\boldsymbol{F} = -\mu P\frac{\boldsymbol{u}}{u}, \tag{1}$$

where $\boldsymbol{F}$ is the friction force, P is the weight of the ball, μ is the coefficient of friction and $\boldsymbol{u}$ is the velocity of the point of contact which is tangent to the supporting plane. This system was first investigated in 1758 by Johann Euler (a son of Leonhard Euler) [25]. We list the basic properties of motion which are important in what follows:

[1] In particular this problem was discussed by the authors of this paper and professor Andy Ruina in the course of the IUTAM Symposium http://iutam2012.rcd.ru/

1. $\boldsymbol{e} = \frac{\boldsymbol{u}}{u} = \text{const}$, i.e. the direction of sliding does not change;
2. if the initial velocity of the center of the ball $\boldsymbol{v}_0$ is not collinear to $\boldsymbol{e} = \frac{\boldsymbol{u}}{u}$, then the center of the ball moves in a parabola (until the ball stops sliding)

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \tfrac{1}{2} f t^2 \boldsymbol{e}, \quad f = \mu g, \tag{2}$$

   where $\boldsymbol{r}$ is the radius vector of the point of contact, $\boldsymbol{r}_0$ is its initial value (for $t = 0$), t is the duration of the motion and g is the free-fall acceleration;
3. The absolute value of the sliding velocity decreases by the law

$$u = u_0 - f\left(1 + \frac{R^2}{\rho^2}\right)t, \tag{3}$$

   where u_0 is the initial value and R and ρ are the radius of the ball and its radius of gyration, respectively.

These properties allow the trajectory of the center of the ball to be uniquely constructed.

Using (2) and (3) it is not difficult to show qualitative and quantitative changes of trajectories of a sliding ball during the passage from the smoother to the rougher surface of the path. In Fig. 3 the families of trajectories of a sliding bowling ball with different initial linear and angular velocities are shown. At the points r_i the coefficient of friction changes from $\mu_1 = 0.04$ to $\mu_2 = 0.06$, $\mu_3 = 0.8$, $\mu_4 = 0.1$, $\mu_5 = 0.5$. As evident from Fig. 3 the growth of the coefficient of friction leads to an earlier termination of the sliding motion of the ball and to a more significant curling of its trajectory. It is also clear that both quantities strongly depend on initial conditions of the system.

Assuming $u = 0$ in (3), it is not difficult to define (see, for example [9]) the time of sliding of the ball on the surface with the coefficient of friction μ:

$$t = \frac{2u_0}{7\mu g}$$

and to estimate it for different couples of surfaces. For example, for the family (1) the total time of sliding is $t = 2.41$ s for the couple $(\mu_1; \mu_5)$, whereas if the ball slides on a homogeneous surface with μ_1, the total time is $t = 7.14$ s.

3. Equation of the boundary curve between two surfaces

Now assume that the coefficient of friction is variable. Let us calculate the boundary between the surfaces on the path in such a way that parallel families of analogous homogenous sliding balls launched under equal initial conditions (linear and angular velocities) on a horizontal rough plane converge to a predetermined point. Let us write the analytical equation for the curve that is the boundary between the surfaces.

Let us choose a starting segment $[AB]$ (without loss of generality we set $[AB] \in Ox$) and assume that for all trajectories emanating from it the vectors $\boldsymbol{v}_0$ and $\boldsymbol{e}$ are equal and non-collinear (see Fig. 4). According to (2), at $t > 0$ this segment moves uniformly. We take