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On a variant of the Maxwell and Oldroyd-B models within the context of a thermodynamic basis

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ABSTRACT

In this paper we develop models within a thermodynamic standpoint that are very similar in form to the classical Maxwell and Oldroyd-B models but differ from them in one important aspect, the manner in which they unload instantaneously from the deformed configuration. As long as the response is not instantaneous, the models that are derived cannot be differentiated from the Maxwell and Oldroyd-B models, respectively. The models can be viewed within the context of materials whose natural configuration evolves, the evolution being determined by the maximization of the rate of entropy production of the material. However, the underpinnings to develop the model are quite different from an earlier development by Rajagopal and Srinivasa [8] in that while the total response of the viscoelastic fluid satisfies the constraint of an incompressible material, the energy storage mechanism associated with the elastic response is allowed to be that for a compressible elastic solid and the dissipative mechanism associated with the viscous response allowed to be that for a compressible fluid, the total deformation however being isochoric. The analysis calls for a careful evaluation of firmly held customs in viscoelasticity wherein it is assumed that it is possible to subject a material to a purely instantaneous elastic response without any dissipation whatsoever. Finally, while the model developed by Rajagopal and Srinivasa [8] arises from the linearization of the non-linear elastic response that they chose and leads to a model wherein the instantaneous elastic response is isochoric, here we develop the model within the context of a different non-linear elastic response that need not be linearized but the instantaneous elastic response not necessarily being isochoric.

1. Introduction

In his seminal paper on the dynamical theory for gases, Maxwell [5] proposed the first rate type model to describe the viscoelastic response of materials. He developed a one-dimensional model and though he did not use the mechanical analog of a spring–dashpot system, the one-dimensional model could be described by a linear spring and a linearly viscous dashpot arranged in series (see [1,10]). Maxwell [5] did not derive a proper three-dimensional model and more importantly did not consider the possibility of the fluid being incompressible. It was left to Oldroyd [7] to systematically develop rate type constitutive relations, to consider constraints such as incompressibility, and to describe the response of non-linear fluids that satisfy appropriate invariance requirements. Amongst the many models that he introduced, one that has attracted considerable attention from those conducting research in flows of non-linear fluids is referred to as the Oldroyd-B model, which when a material constant is set to zero reduces to a three-dimensional generalization of the Maxwell model. The one-dimensional form of the Maxwell and Oldroyd-B models can be motivated by appealing to a spring–dashpot analog but such a consideration leads to ambiguities and difficulties that are discussed later in the introduction.

Oldroyd's derivation was from a purely mechanical standpoint and he did not provide a thermodynamic basis for the development of his model. Recently, Rajagopal and Srinivasa [8] developed a thermodynamic framework within which one could systematically develop the response of rate type fluids. The two main ideas that form the basis of the thermodynamic framework of Rajagopal and Srinivasa [8] are the notions that when a dissipative body deforms due to the action of external stimuli, the body's natural configuration (this can be thought of as the configuration

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that the body would attain on the removal of the external stimuli[1]) evolves, and that this evolution of the natural configuration is determined by how the body produces entropy, or to be more precise the evolution of the natural configuration is determined by the maximization of the rate of entropy production (see Fig. 1).

On selecting an appropriate non-negative rate of entropy production function the latter assumption automatically satisfies the demands of the second law of thermodynamics. Also, instead of making constitutive assumptions concerning the stress, a tensor quantity with six scalar components in the case of the stress being symmetric, they were able to determine the form of the stress by making assumptions on the Helmholtz free energy and the rate of entropy production, two scalars associated with the body. Appealing to such a procedure, and assuming that the Helmholtz free energy was that in a non-linear elastic solid, a neo-Hookean solid, and the rate of entropy production was like that of a viscous fluid, Rajagopal and Srinivasa [8] obtained a three-dimensional model, whose elastic response when linearized led to the model referred to as the Maxwell model. Using the same procedure, they were also able to derive many other popular rate type models, including the Oldroyd-B model. In deriving the model, Rajagopal and Srinivasa [8] enforced the condition that all motions that can be undergone by the fluid are isochoric motions as they were interested in developing a model for a fluid that was incompressible.

The viscoelastic fluid models which are developed within the construct of the evolution of the natural configuration of the body implies that viscoelastic response can be considered as a class of elastic response from an evolving set of natural configurations, the changes in the natural configuration being accompanied by entropy production. As the body is incompressible, it is necessary that all the motions taken by the body have to be isochoric, that is, it is necessary that the motion that takes one natural configuration to another, as well as the instantaneous elastic response from the natural configuration have to be isochoric (see Fig. 2). It is possible to develop a model wherein we require that the total motion of the body be isochoric while the motions associated with the dissipative part and that due to the elastic response are not necessarily be individually isochoric. Interestingly, such an assumption leads to a model that exactly resembles the Maxwell and Oldroyd-B models in the representation of the rate type constitutive relation; however the individual responses such as the instantaneous elastic response and the purely dissipative response between the natural configurations are not isochoric.

In this paper, we relax the condition that both the motion associated with the evolution of the natural configuration and the elastic response be isochoric, but require that the total response be isochoric. Under such an assumption, we find that we can associate a Helmholtz potential for a compressible non-linear elastic body and a rate of entropy production with the body, and carry out the same thermodynamic procedure as used by Rajagopal and Srinivasa [8] and obtain the Oldroyd-B model without taking recourse to linearization of the elastic response. Of course, while the final rate type constitutive relation is exactly the same as the model developed by Oldroyd, it does not satisfy the constraint that the instantaneous elastic response be isochoric. However, one could argue that no process can be truly instantaneous, and for all practical purposes, we have a model that is identical to the model developed by Oldroyd [7]. Thus, if the model referred to as an Oldroyd-B fluid is meant to describe a viscoelastic fluid with the caveat that it cannot be subject to motions wherein there is instantaneous isochoric elastic response, then the model that is derived would qualify to describe such a fluid. As far as the mathematical derivation is concerned one need not appeal to a linearization of the elastic response to develop the model. It should however be emphasized that the fully non-linear model obtained by Rajagopal and Srinivasa [8], which after the linearization of the elastic response is identical to the Oldroyd-B model, will meet the constraint of incompressibility in all motions. Since rate type fluids are not usually thought of as being defined by a class of elastic response functions from an evolving set of natural configurations, the evolution of the configurations being determined by the maximization of the rate of entropy production, the issue that all components of the response have to necessarily be isochoric has not been fully recognized. In fact, the development of such models within the context of springs and dashpots is totally incapable of taking such issues into account for in one-dimensional analysis one cannot extend a spring and require it to be incompressible (isochoricity within the context of one dimension implies inextensibility and in fact rigidity).

To understand the issues clearly, let us consider a rod of the viscoelastic material subject to extension. Since the material is incompressible, it will have to contract in the radial direction. Assume that the extension consists in two parts, an extension due to the elastic part ε_s and an extension associated with the dissipative part ε_d. Each of these extensions ought to be such that they are isochoric, that is, associated with each of them is a radial contraction. Now, from the deformed configuration (see Fig. 3), if one instantaneously unloads to the natural configuration corresponding to the deformed configuration, then such an instantaneous deformation has to be isochoric if the fluid is incompressible. This is not the case with the model that is developed in this paper. Since, as mentioned earlier, the possibility of an instantaneous response is a mathematical idealization that is not possible to enforce physically,[2] in virtue of the fact that one can obtain a model that is identical to that of the Oldroyd-B model in all other motions without having to make an approximation such as the linearization of the elastic response, we feel that the model that is developed in this paper is worth documenting.

It is worth emphasizing that the use of the mechanical analog of a spring–dashpot system obscures a very important point, namely such analogs only lead to models for compressible viscoelastic fluids. If the fluid were to be incompressible, then all responses of the fluid would have to be isochoric. In the development of constitutive relations within the context of spring-dashpot models, one associates a strain with each component of the spring–dashpot system. In the case of the Maxwell fluid, we have a spring and a dashpot in series and with each of these we associate a strain. Now, if the body is given an instantaneous elastic response which implies that the dashpot does not undergo any motion,, as the body is incompressible, which in one dimension implies that the body is inextensible, such a deformation is untenable as the spring cannot change its dimension. An instantaneous elastic response will only be possible if the elastic body were compressible, within the context of one dimensions. That is, the three-dimensional generalizations of the spring–dashpot analogs automatically lead to a model wherein the individual components of the system are compressible. One then has to take such a compressible model and enforce the constraint of incompressibility.

The mechanical analog for Maxwell model is depicted in Fig. 4. The Maxwell model in one dimension is defined through the

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[1] The natural configuration that is relevant can depend on the manner in which the external stimuli is removed. We shall not discuss this issue in detail here.

[2] Recently, Průša and Rajagopal [3] have shown that in the case of the generalization of the Burgers model wherein the viscosity and relaxation time depend upon the mean normal stress, one cannot have solutions even in the sense of distributions to problems involving instantaneous elastic response. Mathematical idealizations such as the possibility of instantaneous elastic response might render simple linear equations amenable to analysis due to mathematical techniques such as Laplace Transforms; within the context of truly non-linear problems they are untenable.