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On the use of energy method with element splitting to determine the stability of a constrained elastica

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ABSTRACT

A constrained elastica under edge thrust may have multiple static equilibrium positions. It is in general difficult to determine the stability of these equilibrium positions due to the presence of unilateral constraints. In this paper we propose an energy method for this purpose. The beam is decretized into a series of rigid links connected at the joints by torsional springs. To deal with the unilateral constraints in question, we allow the contact point on the elastica to be slightly different before and after superposing virtual displacements. In order to accommodate this change of contact point we split the link near the boundary point of the contact region into two sub-links. It is noted that certain restrictions must be imposed on the contact point change in order for the total potential to be stationary if the equilibrium position is symmetric. After linearizing the constraint equations, the matrix associated with the second variation of the total potential before and after superposing virtual displacement can be established. From the eigenvalues of this matrix, the stability of the constrained elastica can be determined. One-point-contact and one-line-contact deformations are discussed in detail. Other deformation patterns can be analyzed in a similar manner. This energy method supplements the vibration method proposed earlier by the first author, in which the contact point is allowed to change during vibration.

1. Introduction

The primary goal of the research in constrained elastica is to understand the behavior of a thin elastic strip under edge thrust when it is subject to lateral surface constraints. This is a highly non-linear mechanics problem which very often admits multiple equilibrium configurations for a specified loading condition [1–10]. In order to determine whether a calculated deformation exists in reality, a stability analysis is needed. However, very few methods are available for the stability analysis of constrained elastica. Recently, Chen and his colleagues [11,12] developed a vibration method which takes into account the change of contact points between the elastica and the external constraint during vibration. From the calculated natural frequencies, the stability of the equilibrium positions can be determined.

Constrained elastica without friction is a conservative elastic system. Intuitively, one should be able to use energy method to determine its stability. Conventional energy method states that if the total potential of a conservative mechanical system has a local minimum at a static equilibrium position, then the equilibrium configuration is stable [13,14]. Domokos et al. [4] tried to apply this concept in constrained elastica, but concluded that the conventional energy method is difficult to implement. One of the difficulties of the stability analysis arises from the fact that while the problem is continuous, the relevant functions are non-smooth due to the contact forces. Doraiswamy et al. [15] used a direct search method to find the equilibrium configuration with the global minimum total potential. Although they can find the equilibrium position with the global minimum total potential, there is no way to tell whether the other equilibrium positions have local minimum or not. The above attempts in using energy method to determine the stability of a constrained elastica encountered the same problem; i.e., it is difficult to write the matrix associated with the second variation of the total potential due to the presence of unilateral constraints.

Some researchers tried to extend the conventional variational principle without unilateral constraints to the one with unilateral constraints [16,17]. One of the main issues here is how to define kinematically admissible virtual displacements. In these works, the authors proposed a contact condition saying that the contact area (in equilibrium) cannot penetrate the rigid boundary after virtual displacement (see Eq. (29) of [17]). They did not consider the situation that after superposing a virtual displacement field onto the equilibrium configuration, the elastic structure may

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contact the constraining wall in a slightly different area. In other words, the domain of contact may be different before and after superposing virtual displacement.

In this paper we propose an energy method, which takes into account the change of contact point before and after superposing virtual displacement, to determine the stability property of a constrained elastica. In Section 2 the elastica is discretized by using a series of rigid links connected by torsional springs. In this way the continuous beam is discretized into a finite degree-of-freedom system. The total potential of the loaded elastica can be written in terms of these degrees of freedom. In Section 3 the one-point-contact deformation of a clamped–clamped elastica is studied in detail. After linearizing the constraint equations properly the matrix associated with the second variation of the total potential is established. Symmetric and asymmetric deformations are investigated separately. In Section 4 we extend the analysis to one-line-contact deformation. In Section 5 other deformation patterns which are variations and extensions of the one-point and one-line-contact deformations are discussed briefly. The results are compared with those predicted via vibration method. In Section 6 several conclusions are summarized.

2. Link-spring model

We consider an inextensible beam of length L under edge load and constrained between a pair of parallel walls. The two ends of the beam can be either clamped or pinned. We consider the case when one end A is fixed in space, and the other end B is clamped and allowed to slide without friction along line AB. An xy-coordinate system is fixed at point A. When the edge thrust P^* at end B is beyond the Euler's critical load, the beam will buckle into a curved shape. As a result, the end B moves to the left a distance ΔL^*. A set of parallel plane walls at $y=\pm H^*$ prevents the elastica from deforming freely. Fig. 1(a) shows the case when the buckled beam touches the upper wall at one point. The contact point is not necessarily in the middle. All contacts are assumed to be frictionless.

The continuous beam described above is modeled as a chain of $N-1$ rigid links of length h_i^* $(i=1, \ldots, N-1)$ connected by torsional springs k_i^*. Link h_i^* is between k_i^* and k_{i+1}^*. The shape of the buckled beam is described by the angles θ_i of the rigid links measured counterclockwise from x-axis. The torsional springs are used to represent the bending resistance of the beam. After replacing the curvature $d\theta/ds$ by a backward finite difference,

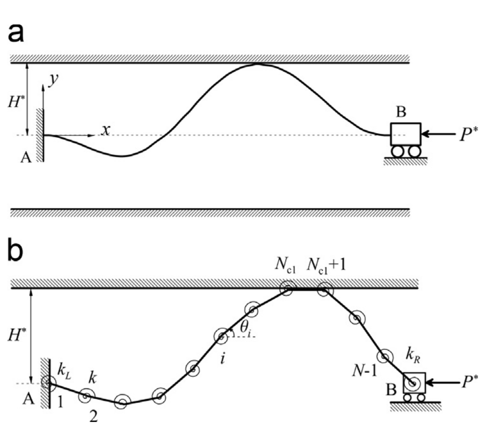

Fig. 1. (a) A clamped buckled beam contacts the upper wall at one point. (b) Link-spring model of a buckled beam in (a).

the bending strain energy of a differential bam element $(EI/2)(d\theta/ds)^2$ can be discretized into $(k_i^*/2)(\theta_i-\theta_{i-1})^2$, where the equivalent spring constant is [18,19]

$$k_i^* = \frac{(EI)_i}{h_i^*} \tag{1}$$

$(EI)_i$ is the bending stiffness of the beam element i. In the following we consider the special case when the beam is uniform along the length. Therefore, k_i^* may be represented by a constant k^*. We also assume that the lengths of the rigid links are equal, so that h_i^* is a constant h^*. The total length $L=(N-1)h^*$. We assume that the discretized beam is supported by spiral springs k_L^* and k_R^* at the ends. In the case of pinned–pinned boundary conditions, both k_L^* and k_R^* are zero. In the case of clamped–clamped boundary conditions, k_L^* and k_R^* are infinity. In practical numerical simulation they are assigned a very large value instead. In this way the continuous beam is discretized into an $(N-1)$-degree-of-freedom system.

The strain energy of the link-spring chain can be written as

$$U_S^* = \frac{k_L^*}{2}\theta_1^2 + \frac{k_R^*}{2}\theta_{N-1}^2 + \frac{k^*}{2}\sum_{i=2}^{N-1}(\theta_i-\theta_{i-1})^2 \tag{2}$$

We assume that the edge thrust is prescribed and increased quasi-statically. This procedure is called load control. The potential corresponding to the external edge load P^* is

$$U_P^* = -P^*h^*\left(N-1-\sum_{i=1}^{N-1}\cos\,\theta_i\right) \tag{3}$$

After introducing the following dimensionless variables (without asterisks),

$$(U_S, U_P, k, k_L, k_R) = \frac{L}{4\pi^2 EI}(U_S^*, U_P^*, k^*, k_L^*, k_R^*),\ P = \frac{L^2}{4\pi^2 EI}P^*,$$
$$(H, h, \Delta L) = \frac{1}{L}(H^*, h^*, \Delta L^*)$$

Eqs. (2) and (3) can written in dimensionless forms

$$U_S = \frac{1}{2}k_L\theta_1^2 + \frac{1}{2}k_R\theta_{N-1}^2 + \frac{1}{2}k\sum_{i=2}^{N-1}(\theta_i-\theta_{i-1})^2 \tag{4}$$

$$U_P = -Ph\left(N-1-\sum_{i=1}^{N-1}\cos\,\theta_i\right) \tag{5}$$

where $k=(N-1)/4\pi^2$ and $h=1/(N-1)$. The total potential of the loaded elastica can be written as

$$U_T = U_S + U_P \tag{6}$$

3. One-point-contact deformation

In the following we take a one-point-contact deformation as an example. Fig. 1(b) shows the discretized model of the buckled beam touching the top wall at point N_{c1}. In modeling point contact at N_{c1}, we assume that the whole link between nodes N_{c1} and $N_{c1}+1$ is on the wall. It is believed that when the number N is large enough, this is a reasonable model for point contact.

First of all, the y coordinate of the right end is zero. Therefore,

$$\sum_{i=1}^{N-1}\sin\,\theta_i = 0 \tag{7}$$