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Vibrations of a lumped parameter mass–spring–dashpot system wherein the spring is described by a non-invertible elongation-force constitutive function

Zhi Yuan [a], Vít Průša [b,1], K.R. Rajagopal [a,*], Arun Srinivasa [a]

[a] Texas A and M University, Department of Mechanical Engineering, 3123 TAMU College Station, TX 77843-3123, United States
[b] Faculty of Mathematics and Physics, Charles University in Prague, Sokolovská 83, Praha 8 – Karlín, CZ 186 75, Czech Republic

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ABSTRACT

The standard setting concerning vibrations of lumped parameter systems is based on the assumption that the mechanical response of the elements of the system is given explicitly in terms of kinematical variables. In particular, the force in a spring element is assumed to be given as a function of the displacement from the equilibrium position. However, some simple mechanical systems such as linear springs with limited compressibility/extensibility do not fit into the standard setting. In this case the displacement must be written as a function of the force. In general, the mechanical response of such elements must be described by an implicit relation between the force and kinematical variables. We study the behaviour of a particular lumped parameter system whose mechanical response is given by a non-invertible expression for the displacement in terms of the force, under harmonic external force. We show that a solution to the original system wherein the displacement is given in terms of the force can be obtained as a limit of a sequence of approximate problems. The approximate problems are designed in such a way that they can be solved using standard numerical methods, and one can avoid using concepts such as set valued mappings. Moreover, we show that the "bounce back" behaviour of the system with linear spring with limited compressibility/extensibility is a direct consequence of the assumed constitutive relation. There is no need to *a priori* supply the rules for the bounce back (impact rules). Further, we show that the advocated approximation procedure is capable of describing the behaviour of the lumped parameter system even in the situations where the governing ordinary differential equation collapses to an algebraic equation. Representative results are demonstrated by a numerical experiment.

1. Introduction

The ordinary differential equation governing the vibration of a spring, dashpot and mass system shown in Fig. 1 takes the form

$$m\frac{d^2x}{dt^2} = -F_s - F_d + F, \tag{1.1}$$

where x is the displacement with respect to the equilibrium position, m is the mass, F denotes the external force, and F_s and F_d denote the forces due to the spring and the dashpot respectively. We shall assume that the external force is given by a harmonic function F. The mechanical properties of the spring and the dashpot are specified via constitutive relations that provide formulae relating the forces F_s and F_d and the kinematical variables x and dx/dt. Traditionally, the mechanical response of the spring and the dashpot are described by (possibly non-linear) equations of the type

$$F_d =_{\mathrm{def}} \mathfrak{F}_d\left(\frac{dx}{dt}\right), \quad F_s =_{\mathrm{def}} \mathfrak{F}_s(x), \tag{1.2}$$

which means that the forces are specified as functions of the kinematical variables. If F_s and F_d are functions of the kinematical variables x and dx/dt, then one can substitute for F_s and F_d into the differential Eq. (1.1), and solve the corresponding initial value problem using standard methods.

* Corresponding author.
E-mail addresses: yuanzhi_master@tamu.edu (Z. Yuan), prusv@karlin.mff.cuni.cz (V. Průša), krajagopal@tamu.edu (K.R. Rajagopal), asrinivasa@tamu.edu (A. Srinivasa).

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In what follows we shall assume that the constitutive relation for the dashpot takes the standard form $F_d =_{\mathrm{def}} c(\mathrm{d}x/\mathrm{d}t)$, while the relation between the force F_s and the position x is given by the following formula

$$x = \begin{cases} -x_{\max}, & F_s \in (-\infty, -F_{s,\mathrm{crit}}), \\ \frac{F_s}{a_1}, & F_s \in [F_{s,\mathrm{crit}}, -F_{s,\mathrm{crit}}] , \\ x_{\max}, & F_s \in (F_{s,\mathrm{crit}}, +\infty), \end{cases} \tag{1.3}$$

where $x_{\max}$ and $F_{s,\mathrm{crit}}$ are given positive constants such that $a_1 =_{\mathrm{def}} F_{s,\mathrm{crit}}/x_{\max}$. A schematic plot of the constitutive relation is given in Fig. 2a. Constitutive relation 1.3 can be thought of as a constitutive relation for a spring with limited extensibility and compressibility.

If the force acting on the spring is in the interval $(F_{s,\mathrm{crit}}, -F_{s,\mathrm{crit}})$, then the spring behaves as a standard linear spring. However, if the spring reaches the maximum possible extension/compression, then the extension/compression cannot be altered by further increase of the magnitude of the applied force. Unlike in the standard setting (1.2), where F_s is given as a *function* of x, the constitutive relation (1.1) has the form $x = f(F_s)$, and the relation $x = f(F_s)$ is *not invertible*. The fact that the force F_s is not a function of the kinematical variable x means that one cannot substitute for F_s and F_d into the differential Eq. (1.1). This makes the problem harder to solve and non-standard tools are needed.

Constitutive relation (1.3) is in fact a special case of a general constitutive relation of type

$$f(F_s, x) = 0, \tag{1.4}$$

where f is a given function. In other words the constitutive relation is specified via an *implicit relation* of the force and the corresponding kinematical variable. This unconventional approach to the description of constitutive relations in the dynamics of lumped parameter systems has been recently advocated by Rajagopal [1], Darbha et al. [2] and Pražák and Rajagopal [3]. In fact, such an approach is necessary if one has to study lumped parameter systems as those considered in this paper as the standard prescription of the constitutive relations for the constituents are impotent to describe them. A detailed discussion of the need as well as the advantages accrued in using such an implicit constitutive relationship for the components of the system can be found in the above mentioned papers and will not be repeated here in detail. (See also Rajagopal [4].) We only briefly illustrate the need for non-standard constitutive relations to describe components of certain lumped parameter systems with the aid of two simple examples of mechanical systems.

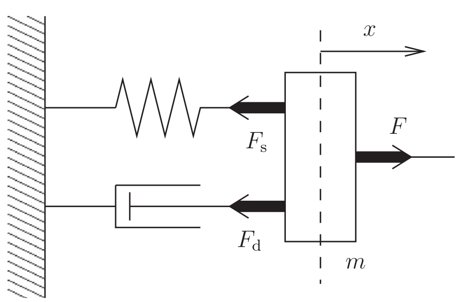

Fig. 1. Spring-dashpot system.

A lumped parameter system such as the one depicted in Fig. 3a cannot be described by providing explicit expression for the force F exerted in the spring in terms of the displacement x. Neither can the shear stress F generated in a viscous dashpot that contains a Bingham fluid, see Bingham [5], be expressed as a function of the shear rate $\mathrm{d}x/\mathrm{d}t$, see Fig. 3b. On the other hand, one can express the displacement of the spring in terms of the force as shown in Fig. 3a, and the shear rate generated in the dashpot can be expressed as a function of the force in the dashpot, see Fig. 3b. Naturally, one could have more complicated situations wherein one might be unable to express explicitly either the force or the displacement (or velocity) in terms of the other. In such a case a truly implicit relation is necessary.

Dynamics of lumped parameter systems wherein the spring and the dashpot are described by constitutive descriptions for the kinematics in terms of the forces have been recently studied by Rajagopal [1] and Darbha et al. [2]. Later, Pražák and Rajagopal [3] proved global existence of solutions to the differential-algebraic system governing the oscillations of certain types of spring–dashpot systems. In this study, we continue the earlier efforts and focus our attention on a lumped parameter system that is portrayed in Fig. 1 with mechanical response of the spring described by constitutive relation shown in Fig. 2a which is a case not covered in the works that have been referenced in this paper.

The fact that we interpret the relation between x and F_s as a *constitutive relation* for the spring cannot be overestimated. The constitutive relation should provide one a well specified relation between the force acting on the spring and the displacement with respect to the equilibrium position. In this respect the interpretation of the constitutive relation shown in Fig. 2b is much better than the interpretation shown in Fig. 2a. Given the force, we have

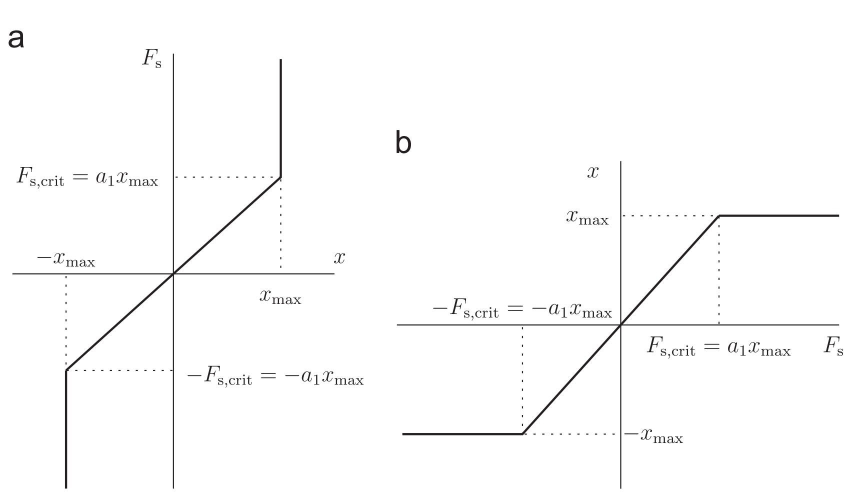

Fig. 2. Constitutive relation for the spring. (a) Force versus position. (b) Position versus force.