International Journal of Non-Linear Mechanics 72 (2015) 1–5

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Planar oscillations of a dumb-bell of a variable length in a central field of Newtonian attraction. Exact approach

Alexander A. Burov [a], Ivan I. Kosenko [b,*]

[a] Dorodnicyn Computing Center of the Russian Academy of Sciences & Higher School of Economics, Russian Federation
[b] Moscow State Technical University of Radio Engineering, Electronics, and Automation, Russian Federation

ARTICLE INFO

Article history:
Received 13 February 2014
Received in revised form
8 January 2015
Accepted 30 January 2015
Available online 7 February 2015

Keywords:
Orbital dynamics
Dumb-bell of a variable length
Newtonian attraction
Relative equilibria
Keplerian orbit
Necessary conditions of stability

ABSTRACT

Dynamics of a dumb-bell of a variable length in a central field of Newtonian attraction is considered. It is assumed that the body moves in a plane fixed in the absolute space and passing through an attracting center. The law of length's variation providing an existence of stationary configurations is pointed out. For these configurations the dumb-bell forms a constant angle with a local vertical passing through the center of mass of the dumb-bell, which moves in an elliptic orbit similar to the Keplerian. In particular, the mentioned constant angle may be equal to zero. In contrast to previous investigations (Burov and Kosenko, 2011 [8,10], Burov, 2011 [9]) the problem is solved within the exact formulation, without supplementary simplifying assumptions concerning smallness of the dumb-bell in comparison to its distance from the attracting center.

1. Introduction

Investigation of orbital systems with a variable mass distribution arises to early 1960s. In particular, a rule for the mass redistribution allowing to keep fixed a direction to the attracting center in the axes fixed in the body was proposed by V.A. Sarychev and W. Schiehlen within a so-called "satellite approximation" (see also [27–30,20]). For the dumbbell-like body of variable length necessary conditions of stability of the radial configuration were studied in [25]. These results were rediscovered and partially completed in [9,10]. Another aspect of the orbital dynamics of bodies with a variable mass distribution relates to possibility of using this redistribution for variation of orbital parameters. Its investigation arising from [4,13] (see also [6]) is a subject of some modern publications (cf. [17,2,14]). The third aspect relates to problems of deployment – retrieval of orbital tethered systems with tethered elements of a finite mass (cf. [11,12,22]). Finally there are investigations devoted to using of dumbbell-like bodies of a variable length for verification of relativistic hypothesis [19,7].

2. Posing a problem

Consider plane motions of a dumbbell-like body P_1P_2 of the length ℓ in a central field of Newtonian attraction. Assume that the masses m_1 and m_2 being located at the endpoints of the dumbbell P_1 and P_2 respectively. Denote as O an attracting center, and as C a center of mass of the dumb-bell. Position of the center of mass is given by polar coordinates (r, ν), where $r = \left|\overrightarrow{OC}\right|$, and ν is an angle between an axis passing through the attracting center, and the vector $\overrightarrow{OC}$. Denote by φ the angle between the vector $\overrightarrow{OC}$ and the dumb-bell, see Fig. 1. Suppose the dumb-bell is subjected by the force F, such that the relation

$$f(\ell, r) = \ell - \ell(r) = 0, \tag{1}$$

holds true in all time of motion. It means that the length of the dumb-bell depends only on the distance between its center of mass and the attracting center.

Kinetic energy of the system reads

$$T = \tfrac{1}{2}\left[m\left(\dot{r}^2 + r^2\dot{\nu}^2\right) + \mu\left(\dot{\ell}^2 + \ell^2(\dot{\nu} + \dot{\varphi})^2\right)\right],$$

$$m = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{2}$$

Since the distances OP_1 and OP_2 remain invariable under reflection of the system with respect to the ray OC, then the potential energy reads

$$U = -GM\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right) = U(r, c), \quad c = \cos\varphi,$$

$$\rho_1 = \left(r^2 - 2\mu_1 \ell r c + (\mu_1 \ell)^2\right)^{1/2}, \quad \mu_1 = \frac{m_2}{m_1 + m_2},$$

* Corresponding author.
E-mail addresses: aburov@ccas.ru (A.A. Burov), kosenko@mirea.ru (I.I. Kosenko).

http://dx.doi.org/10.1016/j.ijnonlinmec.2015.01.011
0020-7462/© 2015 Published by Elsevier Ltd.

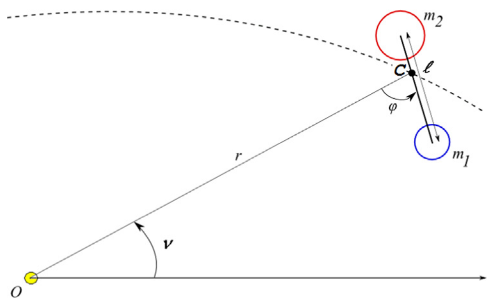

Fig. 1. A dumb-bell in the orbital plane.

$$\rho_2=\left(r^2+2\mu_2\ell rc+(\mu_2\ell)^2\right)^{1/2},\quad \mu_2=\frac{m_1}{m_1+m_2}. \tag{3}$$

Then the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot x}=\frac{\partial L}{\partial x},\quad x\in\{r,\nu,\varphi\}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot\ell}=\frac{\partial L}{\partial \ell}+F,\quad L=T-U \tag{4}$$

can be presented explicitly as

$$m\ddot r=mr\dot\nu^2-\frac{\partial U}{\partial r}, \tag{5}$$

$$\frac{d}{dt}[mr^2\dot\nu+\mu\ell^2(\dot\varphi+\dot\nu)]=0, \tag{6}$$

$$\frac{d}{dt}[\mu\ell^2(\dot\varphi+\dot\nu)]=\frac{\partial U}{\partial c}s,\quad s=\sin\varphi. \tag{7}$$

$$\mu\ddot\ell=\mu\ell(\dot\varphi+\dot\nu)^2-\frac{\partial U}{\partial \ell}+F. \tag{8}$$

Since for $x\in\{r,c,\ell\}$

$$\frac{\partial U}{\partial x}=\frac{\partial U}{\partial \rho_1}\frac{\partial \rho_1}{\partial x}+\frac{\partial U}{\partial \rho_2}\frac{\partial \rho_2}{\partial x},$$

$$\frac{\partial U}{\partial \rho_1}=GM\frac{m_1}{\rho_1^2},\quad \frac{\partial U}{\partial \rho_2}=GM\frac{m_2}{\rho_2^2},$$

$$\frac{\partial \rho_1}{\partial r}=\frac{r-\mu_1\ell c}{\rho_1},\quad \frac{\partial \rho_2}{\partial r}=\frac{r+\mu_2\ell c}{\rho_2},$$

$$\frac{\partial \rho_1}{\partial c}=\frac{-\mu_1\ell r}{\rho_1},\quad \frac{\partial \rho_2}{\partial c}=\frac{\mu_2\ell r}{\rho_2},$$

$$\frac{\partial \rho_1}{\partial \ell}=\mu_1\frac{-rc+\mu_1\ell}{\rho_1},\quad \frac{\partial \rho_2}{\partial \ell}=\mu_2\frac{rc+\mu_2\ell}{\rho_2},$$

then

$$\frac{\partial U}{\partial r}=GM\left[r\left(\frac{m_1}{\rho_1^3}+\frac{m_2}{\rho_2^3}\right)+\mu\ell c\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)\right], \tag{9}$$

$$\frac{\partial U}{\partial c}=GM\mu\ell r\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right), \tag{10}$$

$$\frac{\partial U}{\partial \ell}=GM\mu\left[rc\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)+\ell\left(\frac{\mu_1}{\rho_1^3}+\frac{\mu_2}{\rho_2^3}\right)\right]. \tag{11}$$

To determine a force F allowing to realize relation (1), differentiate twice this relation with respect to time and substitute these expressions for derivatives $\ddot r$, $\ddot\ell$ from (5) and (8). Then

$$\dot\ell=\frac{d\ell}{dr}\dot r,\quad \ddot\ell=\frac{d^2\ell}{dr^2}\dot r^2+\frac{d\ell}{dr}\ddot r,$$

by consequence,

$$\mu^{-1}\left[\mu\ell(\dot\varphi+\dot\nu)^2-\frac{\partial U}{\partial \ell}+F\right]=\frac{d^2\ell}{dr^2}\dot r^2+\frac{d\ell}{dr}\left(r\dot\nu^2-m^{-1}\frac{\partial U}{\partial r}\right)$$

and finally

$$F=\frac{\partial U}{\partial \ell}-\mu\ell(\dot\varphi+\dot\nu)^2+\mu\left[\frac{d^2\ell}{dr^2}\dot r^2+\frac{d\ell}{dr}\left(r\dot\nu^2-m^{-1}\frac{\partial U}{\partial r}\right)\right]. \tag{12}$$

The force F realizing the so-called servoconstraint is not conservative.

Remark 1. The theory of systems subjected to servoconstraints arises to investigations of H. Béghin [3]. There exist numerous publications devoted to the development of this theory, in particular [24,26,16,15].

3. Area integral and change of time, according to Binet–Levi-Civita–Nechvile

The coordinate ν is cyclic. Hence, by virtue of Eq. (6) a function

$$mr^2\dot\nu+\mu\ell^2(\dot\varphi+\dot\nu)=C \tag{13}$$

is a first integral of equations of motion. Here and below the constant C is assumed being positive. Expression (13) can be written in the differential form:

$$mr^2\,d\nu+\mu\ell^2(d\varphi+d\nu)=C\,dt. \tag{14}$$

This allows to represent a speed of variation of the angle ν as

$$\dot\nu=\frac{C}{mr^2+\mu\ell^2+\mu\ell^2\varphi'},\quad (\,)'=\frac{d}{d\nu}. \tag{15}$$

For the motions satisfying condition

$$mr^2+\mu\ell^2+\mu\ell^2\varphi'>0, \tag{16}$$

the angle ν can be used as an independent variable. Often this condition is fulfilled. In particular, it is fulfilled for the motions, such that $|\varphi'|\ll 1$. Then Eqs. (5) and (7) can be represented as

$$m\dot\nu\frac{d}{d\nu}[\dot\nu r']=mr\dot\nu^2-\frac{\partial U}{\partial r}, \tag{17}$$

$$\dot\nu\frac{d}{d\nu}[\mu\ell^2\dot\nu(\varphi'+1)]=\frac{\partial U}{\partial c}s, \tag{18}$$

where the quantity $\dot\nu$ is to be replaced by its value from (15).

Remark 2. The idea of using a true anomaly as an independent variable in problems of the orbital dynamics, turned out being fruitful, arises to investigations of Jacques Philippe Marie Binet, where, as is known, it was used for investigation of motion of a massive point in a central field, in particular, for integration of equations of motion in the Kepler problem. Further it was used by Levi-Civita in the three-body problem [18] and by Nechvile [23] in the restricted elliptic three-body problem (see also [21]). Starting, probably, from publications of Beletsky [5], the true anomaly is effectively used in problems of the attitude motions of satellites in an elliptic problem. The idea of such usage is mainly based on the observation within the satellite approximation independence of motion of the center of mass upon body's motion about its center of mass.

4. Main assumption

Suppose $\ell=\lambda r$. Then

$$\dot\nu=\frac{C}{r^2\left(m+\mu\lambda^2(1+\varphi')\right)},\quad (\,)'=\frac{d}{d\nu}. \tag{19}$$