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Symmetry reductions and exact solutions to the ill-posed Boussinesq equation ☆

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ABSTRACT

In this paper, using the Lie symmetry analysis method, we study the ill-posed Boussinesq equation which arises in shallow water waves and non-linear lattices. The similarity reductions and exact solutions for the equation are obtained. Then the exact analytic solutions are considered by the power series method, and the physical significance of the solutions is considered from the transformation group point of view.

1. Introduction

In this paper, we study the ill-posed Boussinesq equation

$$u_{tt}=u_{xx}+(u^2)_{xx}+u_{xxxx}. \tag{1}$$

This equation describes propagation of long waves in shallow water under gravity [1], in one-dimensional non-linear lattices and in non-linear strings [2]. It should be noted that the well-posed Boussinesq equation differs from Eq. (1) only in the sign of the term containing u_{xxxx}. A numerical scheme using filtering techniques was applied to solve the ill posed Boussinesq equation [3]. Besides, filtering and regularization techniques were applied to Eq. (1) to control growth of errors and to provide better approximate solutions [4] by introducing the singularly perturbed Boussinesq equation

$$\eta_{tt}=\eta_{xx}+(\eta^2)_{xx}+\eta_{xxxx}+\epsilon^2\eta_{xxxxxx}, \tag{2}$$

where ϵ is a small parameter. The double-series perturbation analysis enabled the recovery of the singularly equation (2) of which weakly non-local solitary wave solutions were constructed by using various analytical and numerical methods [5,6].

A wealth of methods have been developed to find exact physically significant solutions of non-linear partial differential equations (PDEs). Some of the most important methods are the inverse scattering method [7], Hirota bilinear method [8], Darboux and Bäcklund transformations [9], Lie symmetry analysis [10–13], CK method [14,15], etc. In this paper, we will apply the Lie group method for solving the ill-posed Boussinesq equation. As is well known, the Lie group method is a powerful and direct approach to construct exact solutions of non-linear differential equations. Furthermore, based on the Lie group method, many other type of exact solutions of PDE can be obtained, such as the traveling wave solutions, soliton solutions, and so on.

The main purpose of this paper is to apply the Lie group analysis method for dealing with symmetries, symmetry reductions and exact solutions to Eq. (1). The outline of the paper is as follows: in Section 2, we perform Lie symmetry analysis for Eq. (1); in Section 3, we discuss the Lie symmetry groups for Eq. (1); in Section 4, we deal with the similarity reductions for Eq. (1) using Lie group method and provide exact solutions; in Section 5, the exact solutions for the reduced equations are obtained by using the power series method; and in Section 6, we summarize and remark our paper.

2. Lie symmetry analysis of the ill-posed Boussinesq equation

In this section, we perform Lie symmetry analysis for Eq. (1), and obtain its infinitesimal generators, commutation table of Lie algebra.

A symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. Once one has determined the symmetry group of a system of differential equations, a number of applications become available.

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Then one can directly use the defining property of such a group and construct new solutions to the system from known ones.

First of all, let us consider a one-parameter Lie group of infinitesimal transformation

$$x \to x + \epsilon\xi(x,t,u), \tag{3}$$

$$t \to t + \epsilon\tau(x,t,u), \tag{4}$$

$$u \to u + \epsilon\phi(x,t,u), \tag{5}$$

with a small parameter $\epsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}. \tag{6}$$

The symmetry group of Eq. (1) will be generated by the vector field of the form (6). Applying the fourth prolongation $pr^{(4)}V$ to Eq. (1), we find that the coefficient functions $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$ must satisfy the symmetry condition

$$-2u_{xx}\phi - 4u_x\phi^x - (2u+1)\phi^{xx} + \phi^{tt} - \phi^{xxxx} = 0, \tag{7}$$

where ϕ^x, ϕ^{xx}, ϕ^{tt} and ϕ^{xxxx} are the coefficients of $pr^{(4)}V$. Furthermore, we have

$$\phi^x = D_x\phi - u_xD_x\xi - u_tD_x\tau, \tag{8}$$

$$\phi^{xx} = D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt}, \tag{9}$$

$$\phi^{tt} = D_t^2(\phi - \xi u_x - \tau u_t) + \xi u_{xtt} + \tau u_{ttt}, \tag{10}$$

$$\phi^{xxxx} = D_x^4(\phi - \xi u_x - \tau u_t) + \xi u_{xxxxx} + \tau u_{xxxxt}, \tag{11}$$

where D_x, D_t are the total derivatives with respect to x and t, respectively.

Substituting (8)–(11) into (7), equating the coefficients of the various monomials in the first, second and the other order partial derivatives and various powers of u, we can find the following equations for the symmetry group of the ill-posed Boussinesq equation:

$$\begin{aligned} &\xi_t = \xi_u = 0, \quad \xi_x = \tfrac{1}{2}\tau_t, \\ &\tau_x = \tau_u = 0, \quad \tau_{tt} = 0, \\ &\phi = -\tfrac{1}{2}(2u+1)\tau_t. \end{aligned} \tag{12}$$

Solving the above Eqs. (12), we obtain

$$\begin{aligned} &o\xi = \tfrac{1}{2}c_1x + c_3, \\ &o\tau = c_1t + c_2, \\ &\phi = -c_1(u + \tfrac{1}{2}), \end{aligned} \tag{13}$$

where c_1, c_2 and c_3 are arbitrary constants.

Hence the Lie algebra of infinitesimal symmetries of Eq. (1) is spanned by the following vector fields:

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \left(u + \frac{1}{2}\right)\frac{\partial}{\partial u}. \tag{14}$$

It is easy to verify that $\{V_1, V_2, V_3\}$ is closed under the Lie bracket. In fact, we have

$$\begin{aligned} &[V_1,V_1] = [V_2,V_2] = [V_3,V_3] = 0, \\ &[V_1,V_2] = -[V_2,V_1] = 0, \\ &[V_1,V_3] = -[V_3,V_1] = \tfrac{1}{2}V_1, \\ &[V_2,V_3] = -[V_3,V_2] = V_2. \end{aligned} \tag{15}$$

Furthermore, we can compute the adjoint representation of the vector field. For Eq. (1), we have

$$Ad(\exp(\epsilon V_i))V_i = V_i, \quad i = 1,2,3,$$

and

$$\begin{aligned} &Ad(\exp(\epsilon V_1))V_2 = V_2, \quad Ad(\exp(\epsilon V_1))V_3 = V_3 - \frac{\epsilon}{2}V_1, \\ &Ad(\exp(\epsilon V_2))V_1 = V_1, \\ &Ad(\exp(\epsilon V_2))V_3 = V_3 - \epsilon V_2, \quad Ad(\exp(\epsilon V_3))V_1 = e^{\frac{\epsilon}{2}}V_1, \\ &Ad(\exp(\epsilon V_3))V_2 = e^{\epsilon}V_2, \end{aligned}$$

where ϵ is an arbitrary constant.

Based on the adjoint representation of the vector field, we obtain the optimal system of the ill-posed Boussinesq equation as follows:

$$\{V_1, V_2, V_3, V_1 + V_2\}.$$

Remark 2.1. For obtaining optimal system of subalgebras, we use discrete transformations: $E_1 : \bar{t} = -t$, $E_2 : \bar{x} = -x$.

3. Symmetry groups of the ill-posed Boussinesq equation

In Section 2, we have obtained the infinitesimal symmetries of Eq. (1). Furthermore, for Eq. (1), the one-parameter groups G_i generated by the V_i for $i = 1,2,3$ are given in the following:

$$\begin{aligned} &G_1 : (x,t,u) \to (x+\epsilon, t, u), \\ &G_2 : (x,t,u) \to (x, t+\epsilon, u), \\ &G_3 : (x,t,u) \to (x + \tfrac{1}{2}x\epsilon, t + t\epsilon, u - (u + \tfrac{1}{2})\epsilon), \end{aligned}$$

where ϵ is any real number. We observe that G_1 is a space translation, G_2 is a time translation, G_3 is a genuinely local group of transformation. They are very important for us to study the exact solutions of PDEs.

Consequently, if $u = f(x,t)$ is a solution of Eq. (1), then $u_{(i)} (i = 1,2,3)$ as follows are solutions of Eq. (1) as well

$$u_{(1)} = f(x - \epsilon, t), \tag{16}$$

$$u_{(2)} = f(x, t - \epsilon), \tag{17}$$

$$u_{(3)} = f(xe^{\epsilon}, te^{-2\epsilon})e^{-2\epsilon} - \tfrac{1}{2}, \tag{18}$$

where ϵ is any real number.

4. Symmetry reductions and exact solutions of the ill-posed Boussinesq equation

Now we deal with the exact solutions for Eq. (1) based on the symmetry analysis. To do this, a linear combination of infinitesimals is considered and its corresponding invariants are determined.

(i) For the generator $V_1 = \partial/\partial x$, we have the following similarity variables:

$$\xi = t, \quad \omega = u,$$

and the group-invariant solution is $\omega = f(\xi)$, that is

$$u = f(t). \tag{19}$$

#Substituting (19) into Eq. (1), we obtain the following reduction equation:

$$f'' = 0, \tag{20}$$

where $f' = df/d\xi$. Therefore, Eq. (1) has a solution $u = c_1t + c_2$, where c_1, c_2 are arbitrary constants.

(ii) For the infinitesimal generator $cV_1 + V_2 = c\partial/\partial x + \partial/\partial t$, where c=0, 1, we have the following similarity variables:

$$\xi = x - ct, \quad \omega = u,$$

and the group-invariant solution is $\omega = f(\xi)$, that is

$$u = f(x - ct). \tag{21}$$

Substituting (21) into Eq. (1), we obtain the following reduction equation:

$$2f'^2 + (1 - c^2)f'' + 2ff'' + f^{(4)} = 0, \tag{22}$$

where $f' = df/d\xi$.