International Journal of Non-Linear Mechanics 69 (2015) 45–54

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

# On wave families in a two-layer falling film

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ARTICLE INFO

*Article history:*
Received 21 August 2014
Accepted 9 November 2014
Available online 20 November 2014

*Keywords:*
Thin fluid film
Interface
Non-linear waves
Bifurcations

ABSTRACT

Using an approximate method, families of non-linear steady-traveling periodic waves in a two-layer falling film have been found for the first time. Computed waves have qualitatively similar behavior as that of those found in homogeneous films but the quantitative characteristics of the waves strongly depend on additional similarity parameters in the two-layer films. In particular, the average location of the interface affects the bifurcation scheme of the waves.



## 1. Introduction

Interest in two-layer (and multilayer) films is stimulated by applications, in particular, in technologies providing mass transfer between two liquids. Due to difficulty of the problem, most attention has been paid to linear models and weakly non-linear models.

Long wave instability of the two-layer flow was studied in [1] in the case of equal dynamic viscosities and different densities of the liquids. The asymptotic method developed in [2] was used to analyze the dependence of the neutral curves for the surface mode on the density ratio and the depth ratio. The interface mode was analyzed in [3], and the general case of both modes in flow with viscous stratification was considered in [4] using the same method.

The linear stability at arbitrary values of the similarity parameters was investigated for the first time in [5] where the generalized Orr–Sommerfeld problem was solved numerically. Two unstable modes associated with the free surface and the interface were computed at moderate values of the Reynolds number. It has been shown that the interface mode corresponds to the Rayleigh–Taylor instability depending on the ratio of the liquids' densities in the case of small wall inclination. It was also found that the interface mode is unstable if the less viscous liquid is in the layer adjacent to the free surface, and this mode is stable if this liquid is adjacent to the solid substrate.

Later the Orr–Sommerfeld problem was used in [6] to investigate the interface and surface modes without taking into account both interface and surface tensions, or the surface tension only. In [7,8] the temporal and spatial growth rates were also calculated without considering the interface and surface tensions. In particular, the result of [5] about the absolute instability in the two-layer flow as the Rayleigh–Taylor instability was confirmed.

The limit case of zero value of the Reynolds number and the absence of the surface and interface tension was analyzed in [9] using an asymptotic method. Mechanism of the surface and long interfacial waves was discussed in [10,11] at zero and very low values of the Reynolds number.

In parallel with the papers directly dealing with the gravity-driven two-layer film flow, there are many works dealing with two types of flows, falling films and interface waves, which are relevant to the considered problem.

Film flow down a vertical plane at moderate flow rates, or a falling film, has been considered in numerous experimental and theoretical investigations. Falling films demonstrate a wide variety of flow regimes, which are very sensitive to flow conditions. The first systematic experimental investigations [12] demonstrated the existence of two principal wave types: periodic sinusoidal waves and solitary waves, traveling with constant velocity. These so-called regular waves can take on different shapes, amplitudes and velocities depending on flow conditions.

The principal method of theoretical investigation based on the use of a thin layer approximation was suggested in [13]. The majority of theoretical results used to describe experimental data were reached in the framework of the Kapitza–Shkadov evolution equations derived in [14] by the integral method. In particular, numerous types of steady-traveling waves in the framework of this approximation, see [15–23] and references in these publications, have been computed. Detailed description of the film theory can be found in monographs [24,25].

Another relevant area of research is the interface instability between two viscous flows. This type of instability was first studied in [26] by the asymptotic method [2] in the case of the plane Couette–Poiseulle flow. In a specific case of liquids with equal densities, it was shown that the flow is unstable for any small value of the Reynolds number, and the instability is supplied by either the moving boundary or the pressure gradient. In [27], a parallel flow of two viscous liquids of equal density in infinite domains divided by a flat interface was

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http://dx.doi.org/10.1016/j.ijnonlinmec.2014.11.020
0020-7462/

studied, and it was shown that there exists a short wave instability in the absence of the surface tension. This mechanism of instability is comparatively small, and it can be stabilized by the surface tension. As illustrated in [28], the interface tension should be unrealistically small to observe the interface instability in the unbounded stratified Couette flow. In the case when one of the liquids is bounded by a wall, and another liquid is unbounded, a long wave interface instability was found in [29]. In [30], a weakly non-linear equation modeling the plane Couette–Poiseulle flow of two liquids was derived, and some examples of wave evolution were computed. A special case of the two-layer Couette flow with high dynamic viscosities ratios was modeled in [31] using an evolution equation derived by the integral method. The integral method was also used to model non-linear solitary waves in two-layer plane flows driven by the gravity [32] and the pressure [33]. Some attention was also paid to stability of interface waves in [34] where a gas–liquid waves were studied. Solitary and periodic waves in an interface between two-liquids were observed in a cylindrical Couette flow at high ratio of the dynamic viscosities [35] and a microchannel [36].

In this paper, we use the integral method to find steady-traveling periodic waves in the two-layer film flow.

This paper is organized as follows: in Section 2, the evolution equations are derived to model flows at real-life values of the similarity parameters. In Section 3, the method used to compute steady-traveling waves is given, and examples of the waves are shown. Finally, conclusions are provided in Section 4.

## 2. Evolution equations

### 2.1. Equations and boundary conditions

To model two-layer film flowing down on a vertical wall, the Cartesian coordinate system $(x,y)$ is introduced with the $x$-axis pointed down and the $y$-axis pointed into the film bulk. We assume that both liquids are immiscible, incompressible and viscous, and we will refer the liquid attached to the wall as ‘1’ and the liquid having the free surface as ‘2’.

The flow is described by the full Navier–Stokes equations and relevant boundary conditions for the velocity components $u$ and $v$ corresponding to the axes $x$ and $y$, respectively, the pressure $p$, the first layer thickness $h^{(1)}$ and the film thickness $h^{(2)}$. To formulate the equations and boundary conditions in the dimensionless form, we take film thickness $H_c$ of the waveless flow as the length scale, and the average film velocity $U_c = Q_c/H_c$, where $Q_c$ is the total flow rate of the film, as the velocity scale. Then dimensional variables are converted into a dimensionless form as

$$t \to \frac{H_c}{\kappa U_c} t, \quad \left(x, y, h^{(1)}, h^{(2)}\right) \to H_c\left(\frac{x_\kappa}{\kappa}, y, h^{(1)}, h^{(2)}\right),$$
$$(u, v) \to U_c(u, \kappa v_\kappa), \quad p \to \rho^{(2)} U_c^2 p,$$

where $\kappa$ is the stretching parameter defined below.

The dimensionless Navier–Stokes equations and the problem boundary conditions are written in the following form:

$$\frac{\partial u}{\partial x_\kappa} + \frac{\partial v_\kappa}{\partial y} = 0,$$
$$\frac{\partial u}{\partial t_\kappa} + u\frac{\partial u}{\partial x_\kappa} + v_\kappa \frac{\partial u}{\partial y} = -\frac{1}{\rho_0^{(j)}}\frac{\partial p}{\partial x_\kappa} + \frac{\nu_0^{(j)}}{\kappa\,\mathrm{Re}}\left(\kappa^2\frac{\partial^2 u}{\partial x_\kappa^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{\kappa \mathrm{Fr}^2},$$
$$\kappa^2\left(\frac{\partial v_\kappa}{\partial t_\kappa} + u\frac{\partial v_\kappa}{\partial x_\kappa} + v_\kappa\frac{\partial v_\kappa}{\partial y}\right) = -\frac{1}{\rho_0^{(j)}}\frac{\partial p}{\partial y} + \frac{\kappa^2\nu_0^{(j)}}{\kappa\,\mathrm{Re}}\left(\kappa^2\frac{\partial^2 v_\kappa}{\partial x_\kappa^2} + \frac{\partial^2 v_\kappa}{\partial y^2}\right),$$
$$y = 0: \quad u = 0, \quad v_\kappa = 0,$$
$$y = h^{(1)}(x_\kappa, t_\kappa): \quad \frac{\partial h^{(1)}}{\partial t_\kappa} + u\frac{\partial h^{(1)}}{\partial x_\kappa} = v_\kappa, \quad [p_{nn}]_1^2 + \frac{\kappa^2 \sigma_0 \varsigma_\kappa^{(1)}}{\mathrm{We}} = 0,$$
$$[p_{n\tau}]_1^2 = 0, \quad [u]_1^2 = 0, \quad [v_\kappa]_1^2 = 0,$$
$$y = h^{(2)}(x_\kappa, t_\kappa): \quad \frac{\partial h^{(2)}}{\partial t_\kappa} + u\frac{\partial h^{(2)}}{\partial x_\kappa} = v_\kappa, \quad p_{nn} - \frac{\kappa^2 \varsigma_\kappa^{(2)}}{\mathrm{We}} = 0, \quad p_{n\tau} = 0, \tag{1}$$

where the notation $[f]_1^2 \equiv f_2 - f_1$ denotes the jump in quantity $f$ from the value in the first liquid, $f^{(1)}$, to the value in the second, $f^{(2)}$. The boundary conditions in (1) include the normal, $p_{nn}$, and tangential, $p_{n\tau}$, stresses and the curvatures $\varsigma_\kappa$ which are calculated as follows:

$$p_{nn} = -p + \frac{2\kappa^2 \rho_0^{(j)}\nu_0^{(j)}}{\kappa\,\mathrm{Re}}\left[1 + \kappa^2\left(\frac{\partial h}{\partial x_\kappa}\right)^2\right]^{-1}$$
$$\times\left[\left(1 - \kappa^2\left(\frac{\partial h}{\partial x_\kappa}\right)^2\right)\frac{\partial v_\kappa}{\partial y} - \frac{\partial h}{\partial x_\kappa}\left(\frac{\partial u}{\partial y} + \kappa^2\frac{\partial v_\kappa}{\partial x_\kappa}\right)\right],$$
$$p_{n\tau} = \frac{\rho_0^{(j)}\nu_0^{(j)}}{\mathrm{Re}}\left[1 + \kappa^2\left(\frac{\partial h}{\partial x_\kappa}\right)^2\right]^{-1}$$
$$\times\left[\left(1 - \kappa^2\left(\frac{\partial h}{\partial x_\kappa}\right)^2\right)\left(\frac{\partial u}{\partial y} + \kappa^2\frac{\partial v_\kappa}{\partial x_\kappa}\right) + 4\kappa^2\frac{\partial h}{\partial x_\kappa}\frac{\partial v_\kappa}{\partial y}\right],$$
$$\varsigma_\kappa = \left[1 + \kappa^2\left(\frac{\partial h}{\partial x_\kappa}\right)^2\right]^{-3/2}\frac{\partial^2 h}{\partial x_\kappa^2}. \tag{2}$$

The system (1) and (2) contains the following dimensionless parameters:

$$\mathrm{Re} = \frac{U_c H_c}{\nu^{(2)}}, \quad \mathrm{We} = \frac{\rho^{(2)} U_c^2 H_c}{\sigma^{(2)}}, \quad \mathrm{Fr}^2 = \frac{U_c^2}{g H_c},$$
$$\rho_0 = \frac{\rho^{(1)}}{\rho^{(2)}}, \quad \nu_0 = \frac{\nu^{(1)}}{\nu^{(2)}}, \quad \sigma_0 = \frac{\sigma^{(1)}}{\sigma^{(2)}},$$

with $\rho_0^{(2)} = 1$, $\rho_0^{(1)} = \rho_0$, $\nu_0^{(2)} = 1$ and $\nu_0^{(1)} = \nu_0$, where $\rho^{(j)}$ and $\nu^{(j)}$, $j = 1,2$ are the densities and viscosities of the liquids, respectively, $\sigma^{(1)}$ and $\sigma^{(2)}$ are the interface and surface tensions, and $g$ is gravity.

The system (1) and (2) has a solution, denoted by capital letters below, describing the steady waveless flow:

$$y \in [0, H]: \quad U^{(1)} = \frac{\mathrm{Re}}{\nu_0 \mathrm{Fr}^2}\left(a^{(1)} y - \frac{y^2}{2}\right), \quad V^{(1)} = 0, \quad P^{(1)} = 0,$$
$$y \in [H, 1]: \quad U^{(2)} = \frac{\mathrm{Re}}{\mathrm{Fr}^2}\left(a^{(2)} + y - \frac{y^2}{2}\right), \quad V^{(2)} = 0, \quad P^{(2)}(y) = 0,$$

where the coefficients

$$a^{(1)} = \frac{1}{\rho_0} + \left(1 - \frac{1}{\rho_0}\right)H, \quad a^{(2)} = \left(\frac{1+\nu_0}{2\nu_0} - \frac{1}{\rho_0\nu_0}\right)H^2 + \left(\frac{1}{\rho_0\nu_0} - 1\right)H$$

have been used. Then the flow rates in the first layer, $Q^{(1)}$, and the second layer, $Q$, are calculated

$$Q^{(1)} = \int_0^H U^{(1)}\,dy = \frac{\mathrm{Re}\,H^2}{2\nu_0 \mathrm{Fr}^2}\left[\frac{1}{\rho_0} + \left(\frac{2}{3} - \frac{1}{\rho_0}\right)H\right],$$
$$Q = \int_H^1 U^{(2)}\,dy$$
$$= \frac{\mathrm{Re}\,(1-H)}{\mathrm{Fr}^2}\left[\frac{1}{3} + \left(\frac{1}{\rho_0\nu_0} - \frac{2}{3}\right)H + \left(\frac{1}{3} + \frac{1}{2\nu_0} - \frac{1}{\rho_0\nu_0}\right)H^2\right]. \tag{3}$$

Finally, we find the total flow rate in the two-layer film

$$Q^{(2)} = Q^{(1)} + Q = \frac{\varphi\,\mathrm{Re}}{\mathrm{Fr}^2},$$
$$\varphi = \frac{H^2}{2\nu_0}\left[\frac{1}{\rho_0} + \left(\frac{2}{3} - \frac{1}{\rho_0}\right)H\right]$$
$$+ (1-H)\left[\frac{1}{3} + \left(\frac{1}{\rho_0\nu_0} - \frac{2}{3}\right)H + \left(\frac{1}{3} + \frac{1}{2\nu_0} - \frac{1}{\rho_0\nu_0}\right)H^2\right].$$

Since we have taken the average velocity of the waveless flow as the velocity scale, the dimensionless total flow rate is $Q^{(2)} = 1$ and thus $\mathrm{Fr}^2 = \varphi\,\mathrm{Re}$. This relation allows us to eliminate the Froude number, and calculate the scale velocity $U_c = \varphi g H_c^2/\nu^{(2)}$.