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Extreme wave solutions: Parametric studies and wavelet analysis

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ABSTRACT

Wave fields for near homoclinic, single mode rogue-wave solutions of the periodic nonlinear Schrödinger equation are presented. Parameters of candidate solutions are estimated and refined through an eigenvalue solution procedure. An overview of the estimation and refining procedure used by the authors is provided. Solutions are scaled to facilitate experimental implementation. The continuous wavelet transform is used to carry out time–frequency analyses and the results obtained are demonstrative of the dispersion relation as well as the time varying side band energy transfer associated with the Benjamin–Feir instability. The analysis framework and approach used are validated with the Peregrine solution. Other extreme wave solutions are analyzed as well. The framework presented here could serve as a basis for experimental investigations into single mode rogue waves as well as other localizations in wave fields.

1. Introduction and background

In the context of many systems, ranging from the ocean [1,2] to optical fibers [3], extreme waves have been reported. These waves are characterized by extreme amplitudes, which can occur intermittently in time and space [4]. The Benjamin–Feir (modulational) instability [5] has been proposed as one of the mechanisms for rogue-wave formation [6]. Given that the nonlinear Schrödinger equation (NLSE) exhibits the Benjamin–Feir instability, the NLSE has been widely studied as a model for extreme wave behavior. Other models include the Dysthe, Kadomtsev Petviashvili (KP), and Korteweg de Vries (KdV) equations. While many solutions and solution families of the NLSE are known, experimental investigations have been limited. Several groups have carried out experiments, in which rogue waves have been provoked in wave tanks [1,2,7]. The typical focus of these experiments has been only on the rogue-wave amplitude. On the other hand, Onorato et al. [8–10] and Osborne [11] have performed laboratory experiments to investigate the presence of modulational instability and rogue wave modes in random wave spectra.

While infrequent, oceanic rogue waves are gaining attention due to their destructive nature. Limited descriptive quantitative data has been recorded [12]. In this work, the authors present a method to determine the parameters of a family of single mode near homoclinic solutions to the NLSE. The first step in the procedure provides a map of eigenvalues which leads to potential solutions. This eigenvalue map may allow researchers to explore new solutions to the NLSE, including rogue-wave solutions. The solutions are derived from a pre-filtering (predictive) and eigenvalue solving (corrector) procedure for the NLSE, as detailed in a recent effort by the authors [13].

Here, for brevity, only a short review of some of the most significant results related to rogue-wave solutions to the NLSE is provided. Tracy [14] presented some of the first analytic solutions to the NLSE. Akhmediev and Korneev [15] determined a family of single parameter solutions. In studies of the nonlinear Schrödinger equation with periodic boundary conditions, homoclinic (or breather) solutions have been used to model rogue waves [16,17]. Islas and Schober [17] were the first to correlate the likelihood of rogue waves with the closeness to homoclinic solutions. Osborne [18] has contributed significant work on hyperfast modeling of rogue waves and presented an analysis of nonlinear Fourier modes of a wavefield, identifying rogue modes. The results have implications in other domains such as fiber optical cables as detailed in [1,19]. A comprehensive review of past contributions and the state of the art related to rogue waves can be found in several review papers (e.g., [6,20,21]).

In the present work, the authors aim to advance the identification and analysis of rogue waves and soliton behavior of the NLSE. As detailed in a recent effort [13], the authors present a procedure, not available elsewhere in the literature, which allows for the identification of spectral parameters of single mode near homoclinic theta function based solutions to the NLSE. While this solution family is already known, examples of variations in the parameters governing the solutions are not readily available in the

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literature. Furthermore, connections between the features of the numerically generated eigenvalue space and the solutions appearing therein have been seldom highlighted. The eigenvalue map presented in this work provides a quantitatively accurate overview and context of the behavior of the solution space from which these solutions originate. The particular solutions presented here, and the insights provided by the mapping procedure, can substantially enhance the understanding and stimulate further investigations into NLSE solutions and associated rogue waves.

Finally, wavelet analyses of rogue-wave solutions are used to characterize evolution in the time–frequency domain. This analysis helps reveal the dispersion relation; that is, low frequency components are identified to travel faster than high frequency components of rogue waves. Furthermore, the time variation of the energy in the side bands associated with a Benjamin–Feir type instability is presented. This analysis is validated on the Peregrine solution and applied to other single mode rogue-wave solutions. It can be easily applied by other researchers to analyze simulated and experimental results. The Peregrine breather has been demonstrated experimentally in a water tank; however, this fact alone does not prove that hydrodynamic surface wave behavior is governed by the NLSE. In fact, the Peregrine solution is only one of many solutions to the NLSE and other wave equations. In order to more conclusively verify the governing model of surface waves, other NLSE solutions would need to be experimentally studied. The predictive results and analysis detailed in this effort can be used to facilitate such experimental efforts.

The remainder of this paper has been arranged as follows. An overview of the eigenvalue map is provided in the next section, along with illustrations of several features of the map. A solution similar to the Peregrine breather is demonstrated. Following this discussion, solutions quite different from the Peregrine breather are presented. In the next section, for a particular spectral parameter variation, the transition from a rogue wave to an exaggerated wave formation is shown. In the fifth section, solutions with physical scaling similar to water waves in published experimental investigations are presented. Finally, continuous wavelet analysis is applied to several solutions to reveal time–frequency behavior including the dispersion relation and the intermittent transfer of energy among the carrier frequency and side bands. Through the discussion presented in the different sections, the authors expect to demonstrate the utility of the solution generation procedure and analysis. Concluding remarks are collected and presented together at the end.

2. Eigenvalue solution method

2.1. Candidate solutions

A brief overview of the procedure to generate the map of candidate solutions to the NLSE is presented in this section. For further details and examples, the reader is referred to the group's previous effort [13]. The scaled NLSE, a dimensionless equation, takes the form

$$iu_T - u_{XX} + 2\sigma|u|^2 u = 0 \tag{1}$$

where $u(X, T)$ is the complex wave envelope field with periodic boundary condition $u(X, T) = u(X + L, T)$ for $0 \le X \le L$, T is the time, X is the spatial variable, $i = \sqrt{-1}$, and the subscripts indicate the associated partial derivatives. The focusing case requires $\sigma = -1$ while $\sigma = 1$ yields the defocusing case. Space periodic spectral solutions to the NLSE can be described as

$$u(X, T) = A\frac{\Theta(X, T|\tau, \delta^-)}{\Theta(X, T|\tau, \delta^+)}e^{2iA^2T}, \tag{2}$$

where $\Theta(X, T|\tau, \delta^{\pm})$ is a Riemann theta function [14,22,23]. A single unstable mode ($N=2$) is considered by expressing $\Theta(X, T|\tau, \delta^{\pm})$ as a two-dimensional theta function defined as

$$\Theta(X, T|\tau, \delta^{\pm}) = \sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\exp i\left[\sum_{n=1}^{2} m_n K_n X + \sum_{n=1}^{2} m_n \Omega_n T + \sum_{n=1}^{2} m_n \delta_n^{\pm} + \sum_{j=1}^{2}\sum_{k=1}^{2} m_j m_k \tau_{jk}\right] \tag{3}$$

The parameters governing the theta function (K_n, Ω_n, and $\delta^{\pm}$) are defined in terms of A, λ_R, λ_I, ϵ_0, and θ, which are referred to as spectral parameters of the near homoclinic solution. Following the notation used in the literature [11], the spectral parameters are defined as

$$\epsilon_1 = \epsilon_0 e^{i\theta}, \quad \epsilon_2 = \epsilon_1^*, \quad \sigma_1 = 1, \quad \sigma_2 = -1 \tag{4}$$

$$\lambda_1 = \lambda_R + i\lambda_I, \quad \lambda_2 = \lambda_1^* \tag{5}$$

$$K_n = -2\sqrt{A^2 + \lambda_n^2}, \quad \Omega_n = 2\lambda_n K_n \tag{6}$$

$$\delta_n^{\pm} = \pi + i\ln(\lambda_n \mp \tfrac{1}{2}K_n) + i\ln\left(\sigma_n\lambda_n - (-1)^n\tfrac{1}{2}K_n\right) \tag{7}$$

$$\tau_{11} = \frac{1}{2} + \frac{i}{\pi}\ln\left(\frac{K_1^2}{\epsilon_1}\right), \quad \tau_{22} = \frac{1}{2} + \frac{i}{\pi}\ln\left(\frac{K_2^2}{\epsilon_2}\right)$$
$$\tau_{12} = \tau_{21} = \frac{i}{2\pi}\ln\left(\frac{1 + \lambda_1\lambda_2 + \frac{1}{4}K_1K_2}{1 + \lambda_1\lambda_2 - \frac{1}{4}K_1K_2}\right) \tag{8}$$

Initially, the solution space is defined with $A=1$, $\theta = 0$, and $\epsilon_0 \le 0.05$. These parameters are refined in a subsequent step leaving λ_R and λ_I as the only free parameters governing the initial function selection. Potential successful parameter combinations are determined by evaluating the periodicity of $u(X, 0)$ over an interval nL (where $n = 1, 2, 3$) for a given (λ_R, λ_I). The periodicity of all solutions is tested over the entire two-dimensional grid of (λ_R, λ_I) values. The periodicity associated with each combination of (λ_R, λ_I) is estimated independently and thus carried out in parallel via a graphics processing unit implementation, as detailed in the authors' recent work [13]. The resulting map of (λ_R, λ_I) pairs which form periodic $u(X, T)$ functions with $L=12$ is shown in Fig. 1. Periodic functions, and hence parameter values of potential solutions, are displayed in white on the candidate map. The dark regions indicate (λ_R, λ_I) pairs which do not result in periodic functions, and hence are not guaranteed to be solutions. If one eliminates these regions and focuses on the light regions, one is guaranteed to find solutions to the NLSE, enables more efficient parameter space exploration. Map variations with respect to L are described in a later section.

2.2. Eigenvalue solution

Given a particular choice of (λ_R, λ_I) from the map described above, the spectral parameters of a near homoclinic solution can be determined by solving the following spectral eigenvalue problem:

$$\Psi_x = Q(\lambda)\Psi, \quad Q = \begin{pmatrix} -i\lambda & u \\ \sigma u^* & i\lambda \end{pmatrix} \tag{9}$$

The eigenvalue solution procedure closely follows prior work [11,13]. The eigenvalue problem of Eq. (9) is recast as a Floquet problem and a constant potential is used to determine the spectral eigenfunction over a given interval. The monodromy matrix is