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On the stability of vertical constant throughflows for binary mixtures in porous layers

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ABSTRACT

A system modeling fluid motions in horizontal porous layers, uniformly heated from below and salted from above by one salt, is analyzed. The definitely boundedness of solutions (existence of absorbing sets) is proved. Necessary and sufficient conditions ensuring the linear stability of a vertical constant throughflow have been obtained via a new approach. Moreover, conditions guaranteeing the global non-linear asymptotic stability are determined.

1. Introduction

The research concerned with fluid motions in porous media has been widely studied, in the past as nowadays, due to the large applications in the real world phenomena like, for instance, in geophysics, in contaminant transport and underground water flow [1–20]. The models describing the fluid motion in porous media are, in general, reaction–diffusion dynamical systems of P.D.Es, which, as it is well known, play an important role in modeling and studying many phenomena (see, for instance [21–29], and references therein). In particular, convection in porous layers in the presence of vertical throughflows finds relevant applications in cloud physics, in hydrological studies, in subterranean pollution and in many industrial processes [30–46,49,50]. The effect of vertical throughflow on the onset of convection has been considered in many cases, like, for example in a rectangular box [37]; combined with a magnetic field [38]; in a cubic Forchheimer model [39]; when the density is a quadratic function of temperature [40] and under the action of an inclined temperature gradient [41,42]. The present paper is devoted to study the effects of both temperature gradient and salt concentration on the stability of a vertical throughflow. Already in [30,31] the authors consider both the effects. Precisely, the effect of variable thermal and solute diffusivities on the onset of convection for non-constant throughflows has been analyzed in [30]; while in [31] the stability of a vertical constant throughflow in a porous layer, uniformly heated and salted from below, has been investigated for determining conditions ensuring the stability in the L^2-norm.

In the present paper, we will analyze the more destabilizing case of horizontal porous layers uniformly heated from below and salted from above by one salt and, by using a new approach proposed by Rionero [13–16], conditions ensuring the stability of a vertical constant throughflow have been obtained.

The plan of the paper is as follows. Section 2 concerns with the introduction of the mathematical model and the determination of a vertical constant throughflow. The definitely boundedness of solutions (existence of absorbing sets) is proved in Section 3, while, in Section 4, the main boundary value problem is analyzed and the independent unknown fields are determined. Section 5 is devoted to the introduction of the linear auxiliary system, while the non-linear equation governing each Fourier component of the perturbations is introduced in Section 6 (according to the methodology introduced by Rionero [13]). In Section 7 the global non-linear stability of the vertical throughflow is investigated. The paper ends with a section devoted to the final remarks.

2. Preliminaries

Let $Oxyz$ be an orthogonal frame of reference with fundamental unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ ($\mathbf{k}$ pointing vertically upwards). The model

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describing the fluid motion in a horizontal porous layer of depth d, uniformly heated from below and salted from above by one salt, in the Darcy–Oberbeck–Boussinesq scheme, is given by

$$\begin{cases} \nabla p = -\dfrac{\mu}{k}\mathbf{v} - \rho_f g\mathbf{k}, \\ \nabla \cdot \mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = K_T \Delta T, \\ \dfrac{\partial C}{\partial t} + \mathbf{v}\cdot\nabla C = K_C \Delta C, \end{cases} \tag{2.1}$$

where

$$\rho_f = \rho_0[1 - \alpha_T(T - T_0) + \alpha_C(C - C_0)] \tag{2.2}$$

is the fluid mixture density and p is the pressure field, T is the temperature field, $\mathbf{v}$ is the seepage velocity, C is the solute concentration field, μ is the dynamic viscosity, k is the permeability, ρ_0 is the reference density, T_0 is the reference temperature, C_0 is the reference solute concentration, $^-\mathbf{gk}$ is the gravitational acceleration, α_T is the thermal expansion coefficient, α_C is the solute expansion coefficient, K_T is the thermal diffusivity, and K_C is the solute diffusivity.

To (2.1) we append the boundary conditions

$$\begin{cases} T(x,y,0,t) = T_L, \quad T(x,y,d,t) = T_U, \\ C(x,y,0,t) = C_L, \quad C(x,y,d,t) = C_U, \end{cases} \tag{2.3}$$

where T_L, T_U, C_L, C_U are positive constants such that $T_L > T_U$ and $C_L < C_U$ (i.e. the layer is uniformly heated from below and salted from above).

On considering the following dimensionless variables:

$$\begin{cases} \mathbf{x}' = \dfrac{\mathbf{x}}{d}, \quad t' = \dfrac{K_T}{d^2}t, \quad \mathbf{v}' = \dfrac{d}{K_T}\mathbf{v}, \quad p' = \dfrac{k(p+\rho_0 g z)}{\mu K_T}, \\ T' = (T - T_0)\tilde{T}, \quad C' = (C - C_0)\tilde{C}, \\ \tilde{T} = \left(\dfrac{\alpha_T \rho_0 g k d}{\mu K_T (T_L - T_U)}\right)^{1/2}, \quad \tilde{C} = \left(\dfrac{\alpha_C \rho_0 g k d}{\mu K_T L_e (C_U - C_L)}\right)^{1/2}, \end{cases} \tag{2.4}$$

system (2.1), omitting all the primes, reduces to

$$\begin{cases} \nabla p = -\mathbf{v} + (\mathcal{R}_T T - \mathcal{R}_S C)\mathbf{k}, \\ \nabla \cdot \mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \Delta T, \\ L_e\left(\dfrac{\partial C}{\partial t} + \mathbf{v}\cdot\nabla C\right) = \Delta C, \end{cases} \tag{2.5}$$

where $L_e = K_T/K_C$ is the Lewis number and

$$\mathcal{R}_T^2 = \frac{k d \rho_0 \alpha_T g \delta T}{\mu K_T} \quad \text{is the thermal Rayleigh number,}$$
$$\mathcal{R}_S^2 = \frac{L_e k d \rho_0 \alpha_C g \delta C}{\mu K_T} \quad \text{is the solute Rayleigh number,} \tag{2.6}$$

with $\delta T = T_L - T_U (>0)$ and $\delta C = C_U - C_L (>0)$. The boundary conditions (2.3) become

$$\begin{cases} T(x,y,0,t) = (T_L - T_0)\tilde{T}, \quad T(x,y,1,t) = (T_U - T_0)\tilde{T}, \\ C(x,y,0,t) = (C_L - C_0)\tilde{C}, \quad C(x,y,1,t) = (C_U - C_0)\tilde{C}. \end{cases} \tag{2.7}$$

A stationary constant throughflow, solution of (2.5)–(2.7), is given by

$$\begin{cases} \mathbf{v}^* = Q\mathbf{k}, \quad Q = \text{const}, \\ T^*(z) = \dfrac{\mathcal{R}_T(e^{Qz} - 1)}{1 - e^Q} - \tilde{T}(T_0 - T_L), \\ C^*(z) = -\dfrac{\mathcal{R}_S(e^{L_e Q z} - 1)}{L_e(1 - e^{L_e Q})} - \tilde{C}(C_0 - C_L), \\ p^*(z) = p_0^* - Qz + \mathcal{R}_T\int_0^z T^*(\xi)\,d\xi - \mathcal{R}_S\int_0^z C^*(\xi)\,d\xi, \end{cases} \tag{2.8}$$

where p_0^* is a constant.

Setting

$$\mathbf{u} = \mathbf{v} - \mathbf{v}^*, \quad \theta = T - T^*, \quad \Gamma = C - C^*, \quad \pi = p - p^*, \tag{2.9}$$

system (2.5) becomes

$$\begin{cases} \nabla \pi = -\mathbf{u} + (\mathcal{R}_T\theta - \mathcal{R}_S\Gamma)\mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \\ \dfrac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = -f_1(z)w - Q\dfrac{\partial\theta}{\partial z} + \Delta\theta, \\ \dfrac{\partial \Gamma}{\partial t} + \mathbf{u}\cdot\nabla\Gamma = f_2(z)w - Q\dfrac{\partial\Gamma}{\partial z} + \dfrac{1}{L_e}\Delta\Gamma, \end{cases} \tag{2.10}$$

where $\mathbf{u} = (u, v, w)$ and

$$f_1(z) = \frac{\partial T^*}{\partial z} = \frac{Q\mathcal{R}_T e^{Qz}}{1 - e^Q}(<0), \quad f_2(z) = -\frac{\partial C^*}{\partial z} = \frac{Q\mathcal{R}_S e^{L_e Q z}}{1 - e^{L_e Q}}(<0). \tag{2.11}$$

To (2.10) we append the boundary conditions

$$w = \theta = \Gamma = 0 \quad \text{on } z = 0, 1. \tag{2.12}$$

In the sequel, we will assume that

(i) the perturbations $\mathbf{u} = (u, v, w), \theta, \Gamma$ are periodic in the x and y directions of periods $2\pi/a_x$ and $2\pi/a_y$, respectively;
(ii) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$ is the periodicity cell;
(iii) u, v, w, θ, Γ belong to $W^{2,2}(\Omega)$, $\forall t \in \mathbb{R}^+$ and can be expanded in the Fourier series, uniformly convergent in Ω, together with all their first derivatives and second spatial derivatives.

Remark 2.1. Let us observe that when the throughflow tends to zero ($Q \longrightarrow 0$), (2.8) reverts to the conduction solution

$$\mathbf{v}_1^* = 0, \quad T_1^*(z) = -\mathcal{R}_T z - \tilde{T}(T_0 - T_L), \quad C_1^*(z) = \frac{\mathcal{R}_S}{L_e}z - \tilde{C}(C_0 - C_L). \tag{2.13}$$

3. Absorbing sets

Let us denote by

- $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the scalar product and the norm in $L^2(\Omega)$, respectively;
- $f_+(x) = \max\{0, f(x)\}$, $f_-(x) = \max\{0, -f(x)\}$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$.

The following lemmas hold.

Lemma 3.1. *Let* $(\mathbf{u}, \theta, \Gamma) \in [W^{2,2}(\Omega)]^5$ *be a solution of* (2.10)–(2.12). *Then*

$$\theta = \tilde{\theta} + \overline{\theta}, \quad \Gamma = \tilde{\Gamma} + \overline{\Gamma} \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{3.1}$$

with

$$|\tilde{\theta}| \le 1, \quad |\tilde{\Gamma}| \le 1, \tag{3.2}$$

and $\overline{\theta}$, $\overline{\Gamma}$ *decreasing functions of t such that*

$$\begin{cases} \|\overline{\theta}(\cdot, t)\| \le \{\|(\theta - 1)_+\| + \|(\theta + 1)_-\|\}_{t=0} e^{-\eta t}, \\ \|\overline{\Gamma}(\cdot, t)\| \le \{\|(\Gamma - 1)_+\| + \|(\Gamma + 1)_-\|\}_{t=0} e^{-\eta t}, \end{cases} \tag{3.3}$$

where

$$\eta = \pi^2 \min\left\{1, \frac{1}{L_e}\right\}. \tag{3.4}$$

Proof. For the proof we refer to [48]. □