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A non-iterative transformation method for Newton's free boundary problem

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ABSTRACT

In book II of Newton's *Principia Mathematica* of 1687 several applicative problems are introduced and solved. There, we can find the formulation of the first calculus of variations problem that leads to the first free boundary problem of history. The general calculus of variations problem is concerned with the optimal shape design for the motion of projectiles subject to air resistance. Here, for Newton's optimal nose cone free boundary problem, we define a non-iterative initial value method which is referred in the literature as a transformation method. To define this method we apply invariance properties of Newton's free boundary problem under a scaling group of point transformations. Finally, we compare our non-iterative numerical results with those available in the literature and obtained via an iterative shooting method. We emphasize that our non-iterative method is faster than shooting or collocation methods and does not need any preliminary computation to test the target function as the iterative method or even provide any initial iterate. Moreover, applying Buckingham Pi-Theorem we get the functional relation between the unknown free boundary and the nose cone radius and height.

1. Introduction

In book II of Newton's *Principia Mathematica* of 1687 several applicative problems are introduced and solved. There, we can find the first calculus of variations problem, predating the brachistochrone problem of the 1690s, that leads to the first free boundary problem of history. The general calculus of variations problem is concerned with the optimal shape design for the motion of projectiles subject to air resistance, see Edwards [11]. Fig. 1 shows a projectile shaped like a bullet with a nose cone having radius r and height h; here x and y are the Cartesian coordinates. This nose cone is a surface of revolution determined by the plane curve $y = y(x)$. The term "nose cone" refers to the generic shape and not necessarily an actual "cone". On the basis of reasonable assumptions on the air resistance, Newton established that the air resistance force F acting on the projectile moving at a velocity v is given by

$$F = 2k\rho v^2, \qquad (1)$$

where ρ is the density of the air, and the fundamental parameter k, indicated as the drag coefficient, is given by the formula

$$k = \int_0^r \frac{2\pi x}{\left[\dfrac{dy}{dx}(x)\right]^2 + 1}\, dx. \qquad (2)$$

This model is based on the assumption of gas flows as independent movement of non-interacting mass particles, which hit the projectile shape and change their momentum. The absence of particle interactions is in contradiction to laminar and turbulent flow models, which are preferred for dense fluids. Thus, the Newton model is only useful for three occasions: motion in low pressure gas, generating good tasks for the calculus of variation and providing the first example of a free boundary problem.

Usually, given a specific configuration of the bullet researchers compute the reduced drag coefficient $k^* = k/\pi r^2$ instead of k, and in Table 1 we report the results concerning several nose cone shapes investigated by Newton.

It is interesting to realize that a rounded hemisphere and a pointed cone provide the same value $k^* = 1/2$. The paraboloid, with $y(1) = 1$, yields a smaller value. For the conical frustum the function $y(x)$ is a straight segment forming an angle θ with the x-axis. The optimally angled flat-tipped conical frustum offers least air resistance when $\tan 2\theta = 2$. Newton's optimal nose cone is defined by the solution of a free boundary problem, to be discussed and solved numerically in the next sections. Newton's flat-tipped frustum offers less air resistance than all of the simple

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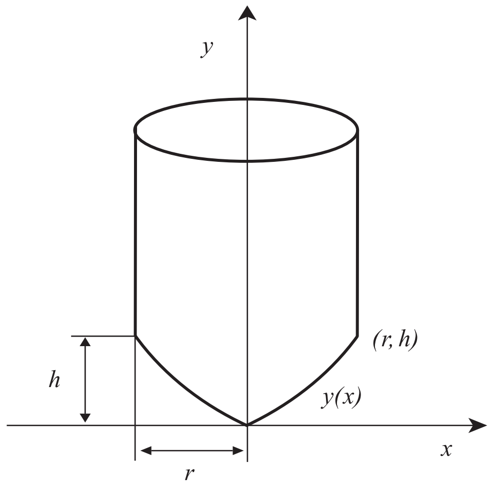

Fig. 1. Physical setup for Newton's projectile shape design.

Table 1
Different nose cone shapes and the corresponding values of the reduced drug coefficient k^* computed for $r=h=1$.

Shape	k^*
Hemisphere	0.5000
Pointed cone	0.5000
Paraboloid	0.4024
Optimal conical frustum	0.3820
Newton's optimal nose cone	0.3748

shapes. There are two competing effects: the flat tip has large air resistance but it allows the nose cone to have steeper sides, which reduces the air resistance. For over 300 years, Newton's solution stood as the minimizer but it is only the radially symmetric flat-tipped minimizer. A radially symmetric nose cone with indented tip results in a smaller value of k^*, that is $k^*=0.29519$, Newton's optimal value, see Gallant [20].

Landau [23] was the first to point out that free boundary problems are always non-linear. Therefore, this kind of problems is often solved numerically. Moreover, normally free boundary problems are transformed into boundary value problems (BVPs), see Ascher et al. [2] or [1, p. 471]. In this context, sometimes, it is possible to solve a given free boundary problem non-iteratively, see the survey reported by Fazio [14], whereas BVPs are usually solved iteratively.

Here, for Newton's free boundary problem optimal nose cone, we define a non-iterative initial value method which is referred in the literature as a non-iterative transformation method (ITM). Indeed, non-ITMs can be defined within Lie's group invariance theory. As far as group invariance theory is concerned, we refer to Bluman and Cole [5], Bluman and Kumei [6], Barenblatt [3], or Dresner [10]. The first application of a non-ITM to a free boundary problem was given by Fazio and Evans [16]. In the past, the main drawback of non-ITMs was that they were considered not widely applicable: see the critical considerations by Fox et al. [19], Meyer [24, pp. 97–98], Na [25, p. 137] or Sachdev [27, p. 218]. In fact, the simplest way in order to verify if a non-ITM is applicable to a particular problem is to use an inspectional analysis as shown by Seshadri and Na [28, pp. 157–168] (cf. also the discussion on inspectional analysis by Birkhoff [4, pp. 99–103]). In relation to the transformation of free boundary problems to initial value problems (IVPs), it is also possible to define an iterative extension of the TM which is always applicable [12,13,15].

2. Newton's free boundary problem

In [11] Edwards exploits modern computer algebra tools to explore the origin and meaning of Newton's nose cone problem. In Fig. 2 we show the physical setup for Newton's optimal flat-tip projectile shape.

Newton's intuition was that the optimal nose cone of least resistance would have a flat circular tip joined to the cylindrical body by a curvilinear band. In this particular case, the value of the reduced drag coefficient can be computed by

$$k^* = a^2 + \int_a^r \frac{2x}{\left[\frac{dy}{dx}(x)\right]^2 + 1}\, dx. \tag{3}$$

But, what should be the radius a of the flat tip and what should be the shape $y=y(x)$ of the arc generating this optimal band by revolution around the y-axis? Nowadays, we would regard this as a calculus of variations problem, complicated by a variable midpoint condition, and proceed to set up the appropriate Euler–Lagrange equation:

$$\left[\frac{\partial}{\partial y} - \frac{d}{dx}\frac{\partial}{\partial \dot{y}}\right]\Phi(x,y,\dot{y}) = 0, \tag{4}$$

where we have used Newton's dot notation for the first derivative and

$$\Phi(x,y,\dot{y}) = \frac{2x}{(\dot{y})^2+1}. \tag{5}$$

Therefore, we get the free BVP

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}\left[\left(\frac{dy}{dx}\right)^2+1\right]}{x\left[3\left(\frac{dy}{dx}\right)^2-1\right]}, \quad x\in[a,r]$$

$$y(a)=0, \quad \frac{dy}{dx}(a)=1, \quad y(r)=h, \tag{6}$$

where the boundary condition on the derivative at $x=a$ means that the tangent to the arc $y(x)$ at a has the same direction of $y=x$, the other two boundary conditions come from the geometrical configuration of the projectile, and a, the length of the frustum, is the unknown free boundary. The second order ordinary differential problem (6) is Newton's optimal free boundary problem.

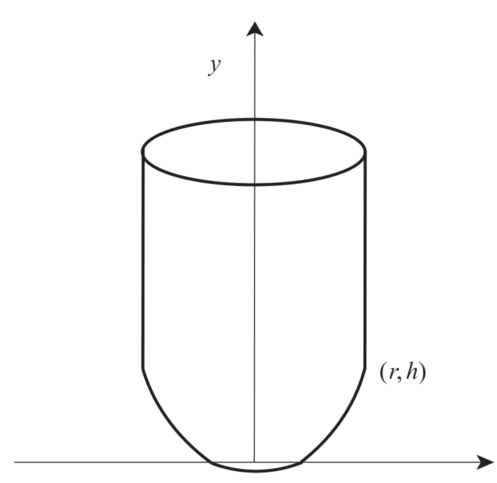

Fig. 2. Physical setup for Newton's optimal flat-tip projectile shape.