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Symmetry classification of first integrals for scalar dynamical equations

K.S. Mahomed*, E. Momoniat

Differential Equations, Continuum Mechanics and Applications, School of Computational and Applied Mathematics, University of the Witwatersrand, Wits 2050, South Africa

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ABSTRACT

We completely classify the first integrals of scalar non-linear second-order ordinary differential equations (ODEs) in terms of their Lie point symmetries. This is performed by first obtaining the classifying relations between point symmetries and first integrals of scalar non-linear second-order equations which admit one, two and three point symmetries. We show that the maximum number of symmetries admitted by any first integral of a scalar second-order non-linear ODE is one which in turn provides reduction to quadratures of the underlying dynamical equation. We provide physical examples of the generalized Emden–Fowler, Lane–Emden and modified Emden equations.

1. Introduction

First integrals are non-constant continuously differentiable functions whose total derivatives vanish identically on the solutions of the corresponding ordinary differential equations. They are quite important in classical mechanics as it deals with second-order systems of equations and constants of the motion. They also arise in other applications as well. More than a century ago, Sophus Lie introduced symmetry-based techniques for solving both partial and ordinary differential equations (ODEs). Lie's approach enables the user to determine Lie groups of symmetries of the given differential equation system. If a sufficiently large symmetry group can be found, it may be used to reduce or solve the underlying differential equation. He proved that the maximum number of point symmetries for scalar nth-order ODEs is $n+4$ if $n \geq 3$ [1]. Lie also showed that scalar first-order ODEs have infinite number of point symmetries. In the case of scalar second-order ODEs Lie proved that the maximum is eight and this is achieved by the free particle equation and indeed scalar second-order ODEs linearizable by means of point transformations. In recent work [2] by Mahomed and Leach they discovered the symmetries of the maximal cases of scalar linear nth-order ODEs, $n \geq 3$. These cases are $n+1, n+2$ and $n+4$. There is another contribution [3] by Leach and Mahomed in which they have found that the Lie algebra of the fundamental first integrals and their quotient of scalar linear second-order ODEs are three-dimensional and have the very interesting property of generating the whole algebra of the equation. This also applies to linearizable by invertible transformations scalar second-order ODEs. The question arises as to what happens for first integrals of scalar second-order ODEs in general. As first integrals are more basic than the equation itself, it is vital to understand their symmetry properties in a general manner.

Lie in [1] obtained the complete classification of scalar second-order ODEs in the complex plane. This was extended to the real plane in [4]. In his book [5], Ibragimov reviews the canonical forms of vector fields of three-dimensional Lie algebras obtained by Lie [1] in the complex domain and makes a comparison with the vectors in the real plane of [4]. For two-dimensional Lie algebras, Ibragimov recalls Lie's results on integrability of scalar second order ODEs. This was earlier mentioned in [4]. The Noether classification of Lagrangians corresponding to scalar second order ODEs that admit low-dimensional Lie algebras was investigated in [6].

In a recent paper [7] we obtained the complete classification of the first integrals of scalar second-order ODEs linearizable by point transformations in terms of their symmetry algebras. It was shown that the maximum symmetry algebra admitted by a first integral of such linearizable second-order ODEs is three. We also obtained a counting theorem which gave the interesting result that a first integral of a scalar second-order linearizable ODE can have 0, 1, 2 or 3 point symmetries [7].

In this work we provide an extension of the classification of the first integrals of scalar second-order ODEs linearizable by point transformations and focus our attention on scalar non-linear second-order ODEs which admit 1, 2 or 3 symmetries. We completely classify

* Corresponding author. Tel.: +27 116804477.
E-mail address: Komalmajeed@hotmail.com (K.S. Mahomed).

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first integrals of scalar non-linear second-order which have submaximal Lie algebras of dimensions 1, 2 and 3. This constitutes new work as symmetries of first integrals of scalar non-linear second-order ODEs which admit 1, 2 or 3 symmetries have not been studied before.

Recall that the first integral

$$I = I(x, y, y'), \tag{1.1}$$

of the scalar second-order ODE $E(x, y, y', y'') = 0$ is said to be invariant under the infinitesimal generator $X = \xi(x, y)\partial/px + \eta(x, y)\,\partial/\partial y$ if and only if

$$X^{[1]}I = 0, \tag{1.2}$$

where X is the first prolonged generator

$$X^{[1]} = X + \zeta_1 \frac{\partial}{\partial y'}, \tag{1.3}$$

with

$$\zeta_1 = D_x(\eta) - y' D_x(\xi), \tag{1.4}$$

in which D_x is the total differentiation operator. Now the reduced first-order ODE $I = C$ also has the symmetry X as is known (see [8]).

This paper is organized as follows. In Section 2, we determine the classifying relations between point symmetries and first integrals of scalar non-linear second-order equations which admit one and two point symmetries. Then we provide the symmetry structure of the first integrals of such non-linear equations. In Section 3, we obtain the classifying relations between symmetries and first integrals of scalar non-linear second-order equations which have three symmetries. There are four equations in Lie's classification that have three symmetries. We investigate each in turn for the symmetry properties of their first integrals. Applications to generalized Emden–Fowler, Lane–Emden and modified Emden equations are given. Concluding remarks are given in Section 4.

2. Non-linear equations with one and two symmetries

In Lie's classification of scalar second-order ODEs in the complex domain [1] there exists one general class of equations with a single point symmetry.

Consider the scalar non-linear second-order ODE in Lie's classification

$$y'' = F(x, y'), \tag{2.1}$$

where F is an arbitrary function. It is indeed easy to see that Eq. (2.1) has in general one point symmetry

$$X = \frac{\partial}{\partial y} \tag{2.2}$$

and the corresponding first integral for (2.1) is

$$I = I(x, y'). \tag{2.3}$$

We observe that the first integral (2.3) has the same symmetry as given in (2.2). Hence the symmetry of the first integral is the same as that of the equation itself.

We next investigate the symmetries of the first integrals of scalar second-order ODEs in Lie's classification which possess two point symmetries. There are two equations, Type I and Type II. We study each in turn for the symmetry structure of their first integrals.

The scalar non-linear equation of Type I

$$y'' = g(y'), \tag{2.4}$$

where g is an arbitrary function of its argument, has two point symmetries

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} \tag{2.5}$$

and the two functionally independent first integrals

$$I_1 = K(y') - x, \quad I_2 = y - y'K(y') + \int K(y')\, dy', \tag{2.6}$$

where

$$K(y') = \int \frac{1}{g(y')}\, dy'.$$

2.1. Classifying relation for the symmetries of the first integrals of $y'' = g(y')$

We let F to be an arbitrary function of I_1 and I_2, viz. $F = F(I_1, I_2)$. Then the symmetry of this general function of the first integrals is

$$X^{[1]}F = X^{[1]}I_1 \frac{\partial F}{\partial I_1} + X^{[1]}I_2 \frac{\partial F}{\partial I_2} = 0, \tag{2.7}$$

where

$$X^{[1]}I_1 = \left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x \frac{\partial}{\partial y'}\right][K(y') - x] = -\xi$$
$$X^{[1]}I_2 = \left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x \frac{\partial}{\partial y'}\right]\left[y - y'K + \int K\, dy'\right] = \eta, \tag{2.8}$$

here ξ, η and ζ_x are given by respectively

$$\xi = a_1, \quad \eta = a_2, \quad \zeta_x = 0. \tag{2.9}$$

These are the coefficients of $X^{[1]}$ which are arrived at by setting

$$X^{[1]} = \sum_{i=1}^{2} a_i X_i^{[1]}, \tag{2.10}$$

where X_i's are the symmetries as given in (2.5) and a_i's are constants. The symmetries of a first integral are always the symmetries of the equation itself. This is a general result proved in [8]. Therefore we have taken here and in what follows $X^{[1]}$ to be a linear combination of the symmetries of the equation under consideration.

After substitution of the values of $X^{[1]}I_1$, $X^{[1]}I_2$ as in (2.8), with ξ, η, ζ_x as in (2.9) and together by using the first integrals $I_1 = K(y') - x$, $I_2 = y - y'K + \int K\, dy'$ in Eq. (2.7), we deduce after some calculations

$$-a_1 \frac{\partial F}{\partial I_1} + a_2 \frac{\partial F}{\partial I_2} = 0. \tag{2.11}$$

The relation (2.11) provides the relationship between the symmetries and the first integrals of the non-linear equation (2.4). We use this to classify the first integrals of (2.4) according to their symmetries.

2.2. Symmetry structure of the first integrals of $y'' = g(y')$

We utilize the classifying relation (2.11) to investigate the number and properties of the symmetries of the first integrals of the ODE (2.4).

For a_1 and a_2 arbitrary, it is seen that the relation (2.11) implies that F is a constant. This clearly means that there is no first integral of (2.4) which has two symmetries (2.5). If one of a_1 or a_2 is arbitrary in (2.11), we end up with either the first integral I_2 or I_1 or an arbitrary function of either. In general we have that $F = F(a_2 I_1 + a_1 I_2)$ has the symmetry $X = a_1 X_1 + a_2 X_2$.

Now we focus our attention on the non-linear equation of Type II

$$xy'' = h(y'). \tag{2.12}$$