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# Non-linear reduced-order model for parametric excitation analysis of an immersed vertical slender rod

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ABSTRACT

A unidirectional three-mode reduced-order model (ROM) for the lateral motion of a slender and immersed rod subjected to harmonic and axial top motion was derived from the continuum equation of motion. Simple trigonometric functions were employed as approximations for the vibration modes and projection functions in Galerkin's method. The non-linear character of the ROM comes from the extensibility of the rod axis and the quadratic hydrodynamic damping. The focus of this investigation is the principal Mathieu's instability with respect to the first vibration mode, i.e., the condition in which the top-motion frequency is twice the structural first natural frequency. Time histories of modal amplitude, as well as maps of post-critical steady-state vibration amplitudes were obtained and discussed. It is seen that, within the principal parametric instability region of the first mode, the time history corresponding to the second classic (sinusoidal) mode oscillates with dominant frequency of the first classic (sinusoidal) mode. Another finding is that, besides the principal Mathieu's instability region, there are also other regions of instability, but with considerably smaller amplitudes. This aspect is due to the non-linear character of the coupled system of equations that defines the ROM.



## 1. Introduction

In the context of a single degree-of-freedom system, the parametric excitation occurs when one of the parameters of the equation of motion (such as, for example, the stiffness of the system) depends explicitly on time. Hill's equation is basically the second-order differential equation of motion of a linear one-degree-of-freedom oscillator, with a time-varying stiffness. In the case in which the stiffness varies sinusoidally with time, Hill's equation becomes Mathieu's equation. Depending on the combination of the average stiffness and the amplitude of its parametric excitation, the trivial solution may be stable or unstable. Strutt's diagram depicts the regions of stability and instability – see, for example, Leipholz [6], Nayfeh and Mook [15], Meirovitch [14] and Xie [26].

Parametric instability is a phenomenon of interest not only to academia but also to practical engineering. This is the case of offshore engineering, particularly in the field of risers dynamics. Risers are slender tubular structures that connect the floating units to the seabed and convey oil and gas. Due to the first-and second-order motions of the floating units. The vessel (rigid) motion is transferred to the riser in a manner that is not detailed in the paper, specially so because both the analytical ROM and the small-scale model used in laboratory experiments are directly excited at the top. Hence, it is essentially an imposed motion at the model top. Of course, such an imposed motion affects directly the normal force and, therefore the geometric stiffness. Hence, Mathieu's instability may occur.

Mathieu's instability in risers and other immersed and slender structures were addressed in several works in the last decades. The pioneer work on the theme is reckoned to be the one by Rainey [20], which demonstrated that the instability zones of Strutt's diagram for the one degree-of-freedom (DOF) tension-leg platform (TLP) model agreed with the available experimental data. Mazzilli [7] studied a “dry model” for the TLP, namely an elastic pendulum with two DOF. This problem, with possible non-linear coupling between modes, is characterized by internal resonance in addition to parametric resonance, which leads to a more complicated behavior than the fundamental one DOF problem. Of course, this scenario is not typical for TLP's. Soares [24] discussed the TLP problem from the standpoint of a single degree-of-freedom model, yet considering the hydrodynamic damping embedded into a linear damping coefficient. Patel and Park [16] investigated Mathieu's instability in the TLP tethers and pointed out that in the unstable region of Strutt's diagram the amplitudes would be limited due to the hydrodynamic damping. The same problem was also investigated by Simos and Pesce [23] considering the variation of tension along the tether and they pointed out the importance of this variation in the parametric instability of tethers.

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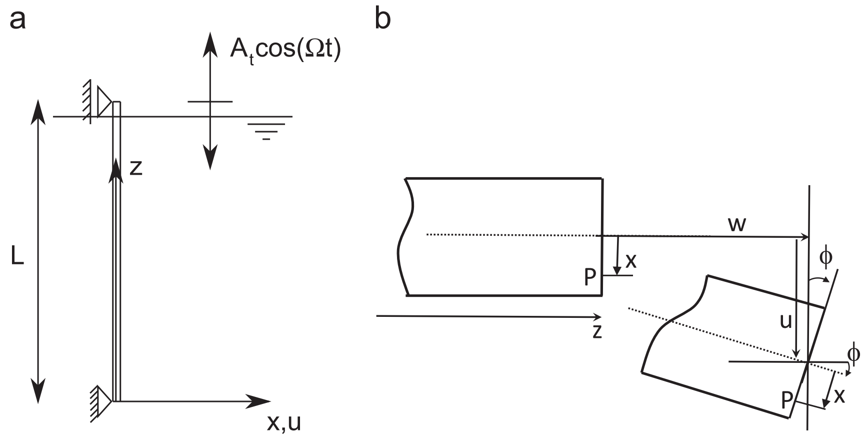


**Fig. 1.** Schematic representations of the flexible rod: (a) Sketch of the vertical rod and (b) Bernoulli–Euler beam model. $\phi$ is the angle between the beam axis in the original and deformed configurations.

Numerical non-linear analysis presented in Zeng et al. [29] showed that if the surge motion of the floating units is larger, the parametric excitation of the risers can be severely affected by this horizontal motion.

It is important to highlight that the mentioned papers consider only the case in which the parametric excitation is due to a harmonic and monochromatic excitation. Yang et al. [28] and Yang and Xiao [27] investigated the problem of multi-frequency parametric excitation of a TTR (top-tension riser) and showed that such a problem may be significantly different from the monochromatic parametric excitation. Yang and Xiao [27] investigated the multi-frequency parametric excitation of TTRs together with the vortex-induced vibration phenomenon. The experimental investigation presented in Franzini et al. [3] focused on the problem of an immersed and slender beam subjected to parametric excitation. Among other results, the authors analyzed Strutt's diagram for some modes, discussing the existence of modal Mathieu's instability.

Analogous to other problems in structural engineering, higher-order models such as those based on the Finite Element Method (FEM) might be used in the analysis of parametric instability (see, for example, [19] and [1]). However, investigations employing higher-order non-linear models can demand larger computational effort. In this way, reduced-order models (ROMs) can offer a qualitative insight of the response with a marked decrease in the number of degrees-of-freedom. Moreover, ROMs are a very useful way to detect complex dynamic phenomena that might otherwise be impossible to investigate in a higher-order model. It is important to highlight that ROMs and FEM must be used as complementary tools aiming at a deeper investigation of the structural dynamics.

The use of ROMs in the dynamic analysis of slender beams subjected to typical riser loading is exemplified in Mazzilli and Poncet [12], who derived a non-linear reduced-order model for the analysis of vortex-induced vibrations of a vertical riser and obtained a good qualitative adherence with FEM results.

A non-linear modal analysis of a slender beam subjected to a linear variation of the axial load (such as in a riser) was carried out by Mazzilli et al. [11]. Analogously to Senjanović et al. [22], the authors embedded the effects of the bending stiffness into a fictitious normal force and found non-linear "Bessel-like" modes. Current use of the "Bessel-like" modes in the analysis of parametric excitation of vertical risers can be found in Mazzilli et al. [13] and Mazzilli and Dias [10].

Herein, we still focus on the problem of a vertical, slender and immersed rod, subjected to parametric excitation due to prescribed harmonic axial top-motions, yet taking into account ROMs with three degrees of freedom, instead of just one, as most of the surveyed studies. Notice that this problem has a counterpart in riser dynamics. The main objective now is to investigate the principal Mathieu's instability region for the first vibration mode, i.e., the condition in which the top-motion frequency is twice the first natural frequency. In Section 2, the three-mode non-linear ROM was derived using sinusoidal functions as "modes" and as projection functions within Galerkin's method. In Section 3, the different results obtained will be analyzed, including a post-critical amplitude map. Finally, Section 4 will present the concluding remarks.

## 2. Three-mode reduced-order model

In order to obtain the system of ordinary differential equations that represents the ROM, it is necessary to firstly obtain the equation of the lateral motion (in one direction) for the continuum vertical rod, with distributed normal force (weight minus buoyancy force) $\gamma$. As the focus of this paper is not the complete derivation of the equation of motion, only the major issues will be addressed. The papers Mazzilli [9], Mazzilli et al. [8] and Mazzilli and Poncet [12] present more detail of the derivation of the equation of motion.

Fig. 1(a) sketches the vertical rod and the system of coordinates. Consider that $m_l$, $EI$, $EA$ and $L_0$ represent the linear mass, the bending stiffness, the axial stiffness and the unstretched length of the rod, respectively. In the static problem, the tension $T(z)$ for the continuum can be written as function of the tension at the top ($T_t$) or at the bottom ($T_b$) through $T(z) = T_t - \gamma L_0 + \gamma z = T_b + \gamma z$.

The sketch of the displacement of the cross-section of a rod is presented in Fig. 1(b). Assuming a Bernoulli–Euler beam model and small rotations, the displacement of a generic point P of the cross-section is given by

$$w_P = w - x\ \sin\ \phi \approx w - x\frac{\partial u}{\partial z} \tag{1}$$

$$u_P = u + x(\cos\ \phi - 1) \approx u \tag{2}$$

where $w$ and $u$ refer to the axial and lateral displacement of the center of gravity of the cross-section, respectively. The strain at this point is given by

$$\epsilon_P = \frac{\partial w}{\partial z} - x\frac{\partial^2 u}{\partial z^2} + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 = \epsilon - x\frac{\partial^2 u}{\partial z^2} \tag{3}$$